On Conformal Submersions and Manifolds with Exceptional Structure Groups

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Abstract

This thesis comes in three main parts. In the first of these (comprising chapters $2-6$), the basic theory of Riemannian and conformal submersions is described and the relevant geometric machinery explained. The necessary Clifford algebra is established and applied to understand the relationship between the spinor bundles of the base, the fibres and the total space of a submersion. O'Neill-type formulae relating the covariant derivatives of spinor fields on the base and fibres to the corresponding spinor field on the total space are derived. From these, formulae for the Dirac operators are obtained and applied to prove results on Dirac morphisms in cases so far unpublished.

The second part (comprising chapters $7-9$) contains the basic theory and known classifications of G_2-structures and $Spin_7^+$-structures in seven and eight dimensions. Formulae relating the covariant derivatives of the canonical forms and spinor fields are derived in each case. These are used to confirm the expected result that the form and spinorial classifications coincide. The mean curvature vector of associative and Cayley submanifolds of these spaces is calculated in terms of naturally-occurring tensor fields given by the structures.

The final part of the thesis (comprising chapter 10) is an attempt to unify the first two parts. A certain '7-complex' quotient is described, which is analogous to the well-known hyper-Kähler quotient construction. This leads to insight into other possible interesting quotients which are correspondingly analogous to quaternionic-Kähler quotients, and these are speculated upon with a view to further research.

Declaration

I do hereby declare that this thesis was composed by myself and that the work described within is my own, except where explicitly stated otherwise.

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Firstly, I would like to thank José Figueroa-O'Farrill and Dmitri Alekseevskii, who supervised my research at different times. Their interest and enthusiasm helped to fuel my own, and this thesis could not exist without them. I have had many useful discussions with José over the past two years and these have been invaluable to my understanding of the subject.

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Contents

List of symbols

Symbol	Meaning
O_n, $\mathfrak{so}_n$	The nth orthogonal group over $\mathbb{R}$ and its Lie algebra
$T\mathcal{M}$	The tangent bundle of the smooth manifold $\mathcal{M}$
$GL(\mathcal{M})$	The full linear frame bundle of $\mathcal{M}$
$\mathfrak{X}(\mathcal{M})$	The space of smooth vector fields on $\mathcal{M}$
ι_X	Interior multiplication by the vector X
π	A Riemannian or conformal submersion
g	Riemannian metric (often on the total space $\mathcal{M}$ of a submersion)
$\mathcal{B}$	The base of a submersion
$\check{g}$	The metric on the base of a submersion
$\mathcal{F}_b$	The fibre of a submersion over the point b of the base
$\hat{g}_b$	The metric on the fibre of a submersion over the point b
e_i	An element of an orthonormal frame on the base $\mathcal{B}$
v_i	An element of an orthonormal frame on a fibre
$\mathscr{V}$	The vertical distribution of a submersion
$\mathscr{H}$	The horizontal distribution of a submersion
v, h	Projection maps to $\mathscr{V}$ and $\mathscr{H}$
$O(\mathcal{M})$	The orthonormal frame bundle of the Riemannian manifold $\mathcal{M}$
$O(\mathcal{M}, \mathscr{V})$	The adapted orthonormal frame bundle with respect to a submersion
$Spin(\mathcal{M})$	The spin structure bundle of the Riemannian spin manifold $\mathcal{M}$
$Spin(\mathcal{M}, \mathscr{V})$	The adapted spin structure bundle
ω, $\hat{\omega}$, $\check{\omega}$	Principal connection forms on various bundles
∇	The Levi-Civita covariant derivative on $\mathcal{M}$
$\hat{\nabla}$, $\check{\nabla}$	Covariant derivatives of the base and fibres of a submersion
$T_X Y$, $A_X Y$	O'Neill's tensor fields
λ^2	The conformal factor of a conformal submersion
Ξ	Induced map of frame bundles $O(\mathscr{H}) \to O(\mathcal{B})$
Ξ^{Spin}	The spin-lift of Ξ
ε	An element or local section of the spin structure bundle
Γ, $\hat{\Gamma}$, $\check{\Gamma}$	Various connection coefficients
d	Exterior differentiation
d^e	Coordinate-wise exterior differentiation with respect to a local frame e
D^π	The horizontal partial derivative, also called absolute derivative or exterior covariant derivative
$C\ell_n$, $\mathbb{C}l_n$	Real and complex Clifford algebras in dimension n
$\mathbb{R}(n)$, $\mathbb{C}(n)$	The algebras of real and complex $n \times n$ matrices
Δ_n	The faithful irreducible representation of $\mathbb{C}l_n$ (or sometimes $C\ell_n$) in even dimensions
$\Delta_n^\pm$	The two faithful irreducibles of $\mathbb{C}l_n$ in odd dimensions
$C\ell_n^0$, $\mathbb{C}l_n^0$	The even parts of the real and complex Clifford algebras in dimension n
Δ_n, $\Delta_n^\pm$	The resulting spaces of semispinors
$\hat{\otimes}$	$\mathbb{Z}_2$-graded tensor product
$\bar{\varphi}$	The involute of the spinor φ
ω_n	The volume element in dimension n

$Spin_n$, $\mathfrak{spin}_n$	The nth spin group over $\mathbb{R}$ and its Lie algebra
$S\mathcal{M}$, $\mathbb{S}\mathcal{M}$	The real and complex spinor bundles of $\mathcal{M}$
ξ	The chosen 'extra' unit vertical vector field
$\mathscr{V}^{\perp\xi}$	The vertical orthogonal complement of ξ
$\varphi^{\perp\xi}$, $\hat{\nabla}^{\perp\xi}$	Various resulting structures dependent on the choice of ξ
$\xi\star$	The illegal action of ξ
$X \cdot \chi$	Clifford multiplication of the vector X and the spinor χ
$\mu^{\mathscr{V}}$	The mean curvature of the vertical distribution $\mathscr{V}$
$\mu^{\mathscr{H}}$	The mean curvature of the horizontal distribution $\mathscr{H}$
$\mathcal{D}$, $\hat{\mathcal{D}}$, $\mathcal{D}^{\mathscr{H}}$	Various Dirac operators
$\mathbb{H}$	The real algebra of quaternions
$\mathbb{O}$	The real algebra of octonions
$\underleftarrow{xyz}$	Another notation for the product $x(yz)$ in $\mathbb{O}$
G_2, $\mathfrak{g}_2$	The Lie group of real automorphisms of $\mathbb{O}$ and its Lie algebra
$\tilde{G}_2$, $\tilde{\mathfrak{g}}_2$	The preimages of G_2 and $\mathfrak{g}_2$ under $Spin_7 \to SO_7$
V	The seven-dimensional space of imaginary octonions $\mathrm{Im}\mathbb{O}$
Δ	The space of real spinors in dimension seven
$\mathfrak{L}$, $\mathfrak{R}$	Left and right octonionic multiplication maps
ϕ_0, ϕ	The canonical G_2-invariant 3-form and corresponding field
φ_0, φ	The canonical G_2-invariant spinor and corresponding field
$*$	The Hodge star operator on an exterior algebra
S^2_0V	The space of symmetric-traceless rank two tensors over V
B_G	A G-structure
$Spin_7^+$, $\mathfrak{spin}_7^+$	The triality-adjusted 7th spin group and its Lie algebra
$\widetilde{Spin}_7^+$, $\widetilde{\mathfrak{spin}}_7^+$	The preimages of $Spin_7^+$ and $\mathfrak{spin}_7^+$ under $Spin_8 \to SO_8$
W	The 'vector' representation of $Spin_7^+$; as vector spaces $W = \mathbb{O}$
Δ^+, Δ^-	The spaces of real semispinors in dimension eight
Φ_0, Φ	The canonical $Spin_7^+$-invariant 4-form and corresponding field
Ψ_0, Ψ	The canonical $Spin_7^+$-invariant spinor and corresponding field
$\delta\phi$	The coderivative of ϕ
$\mathcal{T}$	Either the twistor operator of forms or of spinors
θ	The canonical vector field on manifolds with G_2-structure
T	The canonical symmetric-traceless rank two tensor field on manifolds with G_2-structure
$A \cdot \varphi$	The characteristic tensor field of a G_2-structure acting on φ by Clifford multiplication
F	The orthogonal complement of W in $\Lambda^3 W$
$X \times Y$	The 2-fold cross product of X and Y
$X \times Y \times Z$	The 3-fold cross product of X, Y and Z
$\mathcal{A}$	An associative submanifold
α	The second fundamental form of a submanifold
H	The mean curvature vector of a submanifold
$N\mathcal{A}$	The normal bundle of the submanifold $\mathcal{A}$
$\mathcal{C}$	A Cayley submanifold
Σ	The big space on which the group action and moment map are defined
γ	The tangent vector field to the group action
$\mathcal{M}_\nu$	A level submanifold of the moment map
$\mathcal{B}_\nu$	The quotient of $\mathcal{M}_\nu$ by the group action
E	A Clifford structure
Ω	The fundamental 4-form on a quaternionic-Kähler manifold or the Clifford 4-form of a Clifford structure
∇^E	The covariant derivative induced on E
R^E	The curvature tensor of ∇^E
μ^G	The quaternionic-Kähler or Clifford moment map of a G-action

Conventions

The most important choice of convention in this thesis is that of the embedding of vector spaces $\Lambda V \to T(V)$ where $T(V)$ is the tensor algebra of the vector space V. Two choices are common in the literature and both offer their advantages. We make the choice such that

$$a \wedge b \to \frac{1}{2}(a \otimes b - b \otimes a)$$

for $a, b \in V$, and do not distinguish between elements of ΛV and their images under this map. This convention is used by, amongst many others, Salamon [Sal89], Kobayashi and Nomizu [KN63, KN69] (although they use a different convention for ι_X which changes some formulae) and Sternberg [Ste64] (although he uses a different pairing of ΛV with ΛV^* from the one given by their embeddings in the tensor algebras $T(V)$ and $T(V^*)$, and this effectively means he uses the alternative convention). The alternative to this is to use $a \wedge b \to a \otimes b - b \otimes a$, and this is done by, amongst many others, Besse [Bes87]. An example of one of the most important consequences of this choice is

$$Anti(\nabla) = d = \frac{1}{p+1}\tilde{d}$$

acting on p-forms, where antisymmetrisation of a rank p tensor is defined *with* the factor $\frac{1}{p!}$, and where the $\tilde{d}$ refers to the alternative convention. Also

$$\delta = \frac{1}{p}\tilde{\delta}$$

for δ the coderivative. The Hodge star operator also requires different definitions so that its convenient properties hold:

$$\alpha \wedge *\beta \stackrel{\text{def}}{=} p! g(\alpha, \beta)\omega_{vol} \ , \quad \alpha \tilde{\wedge} \tilde{*}\beta \stackrel{\text{def}}{=} g(\alpha, \beta)\omega_{vol}$$

where α and β are p-forms.

A reasonable convention to adopt is: when a Riemannian metric is fixed, we *do not* distinguish between vector fields and their dual 1-forms. In other words, we consider a finite-dimensional inner product space to be naturally self-dual. For example, on a Kähler manifold or more generally a manifold with almost-Hermitian structure, we consider the complex structure and symplectic form to be the same object and use the same notation for both.

When it is necessary to distinguish between a Clifford representation and a representation of its even subalgebra, we use the notation

$$\Delta_k^0 \stackrel{\text{def}}{=} \Delta_k|_{\mathbb{C}l_k^0} \ .$$

We do not distinguish between the restriction to $\mathbb{C}l_k^0$ and the restriction to $Spin_k \subset \mathbb{C}l_k^0$.

All maps and manifolds in this thesis are smooth, i.e. of class C^∞.

Chapter 1

Introduction

The theory of spinors has had, and continues to have, a great influence on modern mathematics. This has been felt in topology, the most well-known appearances being in index theory and other global applications but here we are more interested in the invaluable role spinors play in *local* geometry, in particular in holonomy theory. A foundational paper is that of Mckenzie Wang [Wan89], in which it is proved

Theorem 1.0.1. *([Wan89]) Let $(\mathcal{M}^n, g)$ be a complete simply connected irreducible Riemannian spin manifold and let N denote the dimension of the space of parallel spinor fields. If $(\mathcal{M}, g)$ is non-flat and $N > 0$ then one of the following holds:*

1. *$n = 2m \geq 4$, $(\mathcal{M}, g)$ has SU_m holonomy and $N = 2$,*
2. *$n = 4m \geq 8$, $(\mathcal{M}, g)$ has Sp_m holonomy and $N = m + 1$,*
3. *$n = 7$, $(\mathcal{M}, g)$ has G_2 holonomy and $N = 1$,*
4. *$n = 8$, $(\mathcal{M}, g)$ has $Spin_7$ holonomy and $N = 1$.*

Conversely, if the holonomy group is one of the above then N must assume the value given.

It is clear how this theorem puts parallel spinors into the framework of the Berger-Simons Holonomy Theorem—undeniably one of the most important results in geometry of the 20th century. It is easy to show that a manifold with non-zero parallel spinor is Ricci-flat (see [Fri00]), and the above theorem may then be used to confirm the results of Bonan [Bon66], that holonomies G_2 and $Spin_7$ occur only on Ricci-flat spaces.

It is often said that the spinor bundle is a refinement of the exterior algebra bundle on a Riemannian manifold. This is justified by an isomorphism between the tensor square of the spinor bundle and the exterior algebra bundle, with some suitable modifications depending on the dimension. Using this 'squaring map' one can use special spinor fields to construct differential forms of geometric interest. A parallel spinor field on a Riemannian manifold satisfying the hypotheses of Theorem 1.0.1 yields parallel differential forms of various degrees. For the four kinds of space of Theorem 1.0.1, the differential forms constructed this way are the Kähler form ω (and its powers) and the complex volume form Ω on a Calabi-Yau manifold, the three Kähler forms ω_1, ω_2, ω_3 on a hyper-Kähler manifold, the associative and coassociative forms ϕ, $*\phi$ on a G_2-manifold and the Cayley form Φ on a $Spin_7$-manifold.

A variation of Theorem 1.0.1 has been proven in the non-simply connected case by Wang [Wan95]. Also see the work of McInnes [McI98] and Moroianu and Semmelmann [MS00] for further details on the existence of parallel spinors (Moroianu and Semmelmann correct a mistake of McInnes). The pseudo-Riemannian cases have been studied by Baum [BK99].

A weaker condition for a spinor field is that of being a *Killing spinor*. This is a spinor field χ satisfying

$$\nabla_X \chi = \mu X \cdot \chi$$

for all vector fields X and constant μ. For complex spinors, the Killing constant μ must be either real or purely imaginary; we say the Killing spinor is real-Killing or imaginary-Killing respectively (to be very precise, we could say e.g. real-Killing complex spinor). If a non-parallel Killing spinor exists then the manifold is locally irreducible and is Einstein (see [Fri00]) with scalar curvature given by $4n(n-1)\mu^2$. For complete metrics, the real-Killing spinors exist only on compact spaces and the imaginary-Killing spinors only on non-compact spaces. In the case of complex spinors, Christian Bär [Bär93] proved the result

Theorem 1.0.2. *([Bär93]) Let $(\mathcal{M}^n, g)$ be a complete simply connected Riemannian spin manifold with non-parallel real-Killing spinor. If $(\mathcal{M}, g)$ is not a round sphere then one of the following holds:*

1. *$n = 2m-1$, $m \geq 3$ odd and $(\mathcal{M}, g)$ is Einstein-Sasaki,*
2. *$n = 4m-1$, $m \geq 2$ and $(\mathcal{M}, g)$ is Einstein-Sasaki but not 3-Sasaki, or is Einstein 3-Sasaki,*
3. *$n = 6$ and $(\mathcal{M}, g)$ is nearly-Kähler,*
4. *$n = 7$ and $(\mathcal{M}, g)$ is nearly-G_2.*

The converse of the theorem also holds but we do not provide details here.

The irreducible hypothesis is not needed for Theorem 1.0.2 because the existence of a non-parallel Killing spinor automatically implies (local) irreducibility.

The method used in [Bär93] to prove Theorem 1.0.2 is the now well-known *cone construction*, which works as follows. If $\mathcal{M} = \mathbb{R} \times_{r^2} \mathcal{B}$ is the Riemannian cone over the complete simply connected Riemannian spin manifold $\mathcal{B}$, there is a natural homomorphism of oriented orthonormal frame bundles $SO(\mathcal{M}) \to SO(\mathcal{B})$. There is a natural spin structure on $\mathcal{M}$ given by that on $\mathcal{B}$: the unique one such that the map of frame bundles is covered by a homomorphism $Spin(\mathcal{M}) \to Spin(\mathcal{B})$. The complex spinor bundles are the bundles associated to $Spin(\mathcal{M})$ and $Spin(\mathcal{B})$ by the complex spin representations Δ^0_{n+1} and Δ^0_n. This allows us to define the *basic lift* of a spinor field on $\mathcal{B}$ to a spinor field on the cone $\mathcal{M}$. The Levi-Civita connections (yielding the spin connections) do not commute with the basic lift operation and instead we find a formula is satisfied by the spinorial covariant derivatives. This tells us that a Killing spinor on $\mathcal{B}$ lifts to a parallel spinor on the cone $\mathcal{M}$ and Theorem 1.0.1 can be applied.

It is interesting that, although $\mathcal{B}$ can be considered a submanifold of the cone $\mathcal{M}$, the vanishing of the covariant derivative on $\mathcal{M}$ does not imply the covariant derivative vanishes along $\mathcal{B}$. This is in stark contrast to the classical Gauss formula for the covariant derivative of a *vector field* along a submanifold.

Note that through the cone construction, the non-spherical hypothesis of Theorem 1.0.2 corresponds precisely to the non-flat hypothesis of Theorem 1.0.1. There are many real-Killing spinors on the round spheres (see [Bär93]), and spheres do not admit Riemannian metrics of non-generic holonomy.

Although we mention Bär's work first, the analgous results for imaginary-Killing spinors actually predate this. Helga Baum [Bau89b, Bau89a] defines two types of imaginary-Killing spinor field on a Riemannian manifold. It is then shown that those of the second type occur only on the hyperbolic spaces H^n. Those of the first kind are more interesting; if $(\mathcal{M}, g)$ is a complete connected Riemannian spin manifold with an imaginary-Killing spinor of the first type then $\mathcal{M} = \mathbb{R} \times_{e^{-2t}} \mathcal{B}$ where $\mathcal{B}$ is a complete connected Riemannian spin manifold with a non-zero parallel spinor field. Theorem 1.0.1 can then be applied. Notice that the warped product relationship is the opposite way around for imaginary-Killing spinors to the case of real-Killing spinors. This time, the space $(\mathcal{M}, g)$ with imaginary-Killing spinors *is* the warped product and we must find the summand $\mathcal{B}$ that admits a parallel spinor. The warping factor is e^{-2t}, so instead of a cone we can visualise $\mathcal{M}$ as a pseudosphere shape and $\mathcal{B}$ as one of the parallel circles.

The method of proof Baum uses for this result is equivalent to our description of Bär's. In this thesis we do not follow the notation of either Bär or Baum as it is not comprehensive enough to allow the amount of detail we need. Bär and Baum do not even write in a similar way to each other, as we can see in the papers [Bär93] and [Bau89b, Bau89a]. The reader is therefore referred to these for further details.

Other constructions belong in the same family as those of Bär and Baum. One of these is the notion of *generalised cylinders* introduced by Bär, Gauduchon and Moroianu [BGM05]. To describe a one-parameter collection $\{g_t | t \in \mathbb{R}\}$ of metrics on a manifold $\mathcal{B}$, the authors define a single metric $g = dt^2 + g_t$ on $\mathcal{M} = \mathbb{R} \times \mathcal{B}$. This is much more general than a warped product and has several interesting applications. When the g_t's are Riemannian, any pair can be joined by a straight line in the convex cone of Riemannian metrics on $\mathcal{B}$. This provides a way to identify the spinor bundles of all the metrics in the family using the spinor bundle of $(\mathcal{M}, g)$. The same problem is considered when the g_t's are pseudo-Riemannian and then things are much more complicated. Interestingly, the generalised cylinder framework can be used to give a new and simple proof of the fundamental theorem of hypersurface theory: given a Codazzi tensor on $\mathcal{B}$, an explicit metric on the cylinder $\mathcal{M} = \mathbb{R} \times \mathcal{B}$ is given which is shown to be flat. Of most interest to us is the use of generalised cylinders in classifying manifolds with *generalised Killing spinors*. These are spinor fields χ satisfying

$$\nabla_X \chi = \frac{1}{4} A_X \cdot \chi$$

for all vector fields X, where A is a symmetric endomorphism field. When A is proportional to the identity these are Killing spinors. Morel [Mor03] has studied the case when A is parallel. Bär, Gauduchon and Moroianu consider the case when A is a Codazzi tensor, and prove that when such a χ exists the manifold can be embedded as a hypersurface into a Ricci-flat space with a parallel spinor whose restriction is χ.

This thesis has been supervised by two different mathematicians, Dmitri Alekseevskii and José Figueroa-O'Farrill, for different periods of the research. As a result the thesis splits into three parts—the first two corresponding to the two supervisors and the third an attempted synthesis of the first two. Let us now discuss the first topic, motivated by Dmitri Alekseevskii.

The Riemannian cone may be viewed as a map $\mathcal{M} = \mathbb{R} \times_{r^2} \mathcal{B} \to \mathcal{B}$. This is a submersion of manifolds with one-dimensional fibres which preserves the conformal structure. This perspective suggests a generalisation of Bär's procedure to arbitrary conformal submersions of Riemannian spin manifolds. Does such a generalisation produce a correspondence of spinorial covariant derivatives which eliminates not a one-dimensional piece (as for Killing spinors and the cone) but other, higher-dimensional components? We do not fully answer this question in this thesis but we do make significant progress towards an answer, namely by deriving relations for spinorial covariant derivatives on the total space $\mathcal{M}$, base space $\mathcal{B}$ and fibres $\mathcal{F}$ of a conformal submersion of Riemannian spin manifolds.

We begin by reviewing the theory of *Riemannian submersions*. These are those maps which preserve both lengths and angles in the horizontal distribution $\mathscr{H}$. The foundational paper of O'Neill [O'N66], as well as the earlier work of Ehresmann, Hermann and Wolf, contains all the relevant material. In particular, O'Neill defines two fundamental tensor fields T and A on $\mathcal{M}$ that, in a sense, characterise the Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$. The field T is the second fundamental form of the fibres and A is the curvature of the submersion, which can be thought of as the second fundamental form of the base (of course, the base is not contained in the total space). We follow this with the generalisation to *conformal submersions*, of which the cone projection is a special case. The fundamental tensors T^g and A^g are defined in the same way as in the Riemannian case and can be related to the tensor fields $T^{\lambda^{-2}g}$ and $A^{\lambda^{-2}g}$ of the associated Riemannian submersion. This time T^g is the obstruction to the fibres being totally umbilic. Some ground-level calculations allow us to write expressions for the coefficients of the Levi-Civita connection of $\mathcal{M}$ in terms of the coefficients of that of $\mathcal{B}$ and of $\mathcal{F}$ as well as parts involving the fundamental tensor fields.

A common problem has been found lying at the heart of the constructions of Bär, Baum and Bär, Gauduchon and Moroianu. This is, how can we identify the spinor bundles of the big space $\mathcal{M}$ and the small space $\mathcal{B}$? Bär and Baum were lucky in that their projection maps are conformal and we can identify the relevant frame bundles (Bär, Gauduchon and Moroianu did not have this property for their generalised cylinders and therefore had to use other means). We are lucky in this way too, but our situation is far more complicated than these others because the fibre $\mathcal{F}$ is not one-dimensional. Higher-dimensional fibres add a whole layer of complication because the spinor bundles of the fibres are now needed to relate the spinor bundles of $\mathcal{B}$ and $\mathcal{M}$. Of course, this is not a problem of spinor bundles but rather one of Clifford representations. There are four cases, given by the parity of $\dim \mathcal{B}$ and $\dim \mathcal{M}$. When both of these numbers are even the relationship between the spin representations is understood (see e.g. [LS09]) and is expressed as a $\mathbb{Z}_2$-graded tensor product. In the other cases we have to find the correct constructions ourselves and we do this by first looking at the behaviour in low dimensions. We are able to understand the Clifford representations in all cases, albeit a very complicated exercise. We should point out that our attention is restricted to complex spinors here, to allow for generality in the dimensions. The real and quaternionic cases can be obtained from the complex one with a bit of patience.

The relations between the spinor bundles of $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{F}$ then follow from those between the appropriate Clifford representations. Once these are understood we can begin to calculate covariant derivatives using the connection coefficient formulae we found earlier. Here we present the main theorem in this part, in the simplest of four cases.

Theorem 1.0.3. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim \mathscr{V}, \dim \mathscr{H} > 1$ with $\dim \mathscr{V}$ and $\dim \mathscr{H}$* ***both even****. Let φ be a vertical spinor field and let χ and ψ be horizontal spinor fields on $\mathcal{M}$, as in the notation of Theorem 5.1.5, and consider the spinor field on $\mathcal{M}$ constructed from these.*

Then for X a horizontal and U a vertical vector field, the covariant derivatives are given by

$$\begin{aligned}\nabla_X(\varphi \mathbin{\hat{\otimes}} \chi) &= (\hat{\nabla}_X \varphi) \mathbin{\hat{\otimes}} \chi + \varphi \mathbin{\hat{\otimes}} (\nabla^{\mathscr{H}}_X \chi) + \frac{1}{4} A^g_X \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ &\quad - \frac{1}{2\lambda} \varphi \mathbin{\hat{\otimes}} X \cdot hgrad^g \lambda \cdot \chi - \frac{X(\lambda)}{2\lambda} \varphi \mathbin{\hat{\otimes}} \chi \ ,\end{aligned}$$

$$\begin{aligned}\nabla_U(\varphi \mathbin{\hat{\otimes}} \chi) &= (\hat{\nabla}_U \varphi) \mathbin{\hat{\otimes}} \chi + \varphi \mathbin{\hat{\otimes}} (\nabla^{\mathscr{H}}_U \chi) + \frac{1}{4} \varphi \mathbin{\hat{\otimes}} A^g U \cdot \chi \\ &\quad + \frac{1}{4} T^g_U \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \ .\end{aligned}$$

It is hoped this theorem will have many applications. The fundamental tensor field A^g can be interpreted as the second fundamental form of the base $\mathcal{B}$ (although $\mathcal{B}$ does not live in $\mathcal{M}$, so not in the usual way) and Theorem 1.0.3 should be compared with the defining formula of generalised Killing spinors. We discuss this further in Chapter 11.

Once we have expressions for covariant derivatives, a natural next step is to calculate expressions for the Dirac operators. This does not require us to concoct new machinery and is a complicated but straightforward calculation. Again presented here in the simplest of four cases, we find

Theorem 1.0.4. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim \mathscr{V}, \dim \mathscr{H} > 1$ with $\dim \mathscr{V}$ and $\dim \mathscr{H}$* ***both even****. Let φ be a vertical spinor field and let χ and ψ be horizontal spinor fields on $\mathcal{M}$, as in the notation of Theorem 5.1.5, and consider the spinor field on $\mathcal{M}$ constructed from these. The Dirac operators of the total space, base and fibres are related by*

$$\begin{aligned}\mathcal{D}(\varphi \mathbin{\hat{\otimes}} \chi) =& (\hat{\mathcal{D}} \varphi) \mathbin{\hat{\otimes}} \chi + \bar{\varphi} \mathbin{\hat{\otimes}} (\mathcal{D}^{\mathscr{H}} \chi) + \frac{1}{8} A^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi) - \frac{p}{2} \mu^{\mathscr{V}} \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ &+ \sum_{i=1}^{p} (v_i \cdot \varphi) \mathbin{\hat{\otimes}} (\nabla^{\mathscr{H}}_{v_i} \chi) + \sum_{i=1}^{n} (\hat{\nabla}_{\lambda^{-1} e_i} \bar{\varphi}) \mathbin{\hat{\otimes}} (\lambda^{-1} e_i) \cdot \chi \\ &- \frac{n-1}{2\lambda} \bar{\varphi} \mathbin{\hat{\otimes}} hgrad^g \lambda \cdot \chi \ .\end{aligned}$$

Such a formula has been published already in [LS09], but only for the case $\dim \mathcal{B}$ is even. Its use there is to characterise so-called *Dirac morphisms*, which are maps that preserve the germ of the Dirac operator. Whilst their definition for $\dim \mathcal{F} > 1$ is cumbersome they are relevant because they are spinorial analogues of maps that preserve the germ of the Laplacian—harmonic Riemannian submersions. Harmonic maps have been studied extensively and we do no more than present some basic definitions and motivation for the theory. More general than harmonic Riemannian submersions are *harmonic morphisms*, which are maps that preserve the kernel of the Laplacian and not necessarily the entire spectrum. We discuss these briefly as well, and apply Theorem 1.0.4 to prove the more difficult analogue of the characterisation of Dirac morphisms proved in [LS09].

This brings to an end the first part of the thesis, although it is hoped many more applications can be found for the formulae found therein. We now discuss the second topic, motivated and supervised by José Figueroa-O'Farrill.

Consider again the list of holonomy groups allowed for Riemannian manifolds admitting a parallel spinor in Theorem 1.0.1. As we remarked, the squaring operation allows us to construct the various parallel differential forms on these spaces. Riemannian manifolds with holonomy SU_n or Sp_n have been studied extensively and the rudiments of these theories can be found in [Bes87] and [Sal89]. Manifolds with restricted holonomy exactly G_2 or $Spin_7$ were later discovered through the efforts of Bryant [Bry87], Bryant and Salamon [BS89] and Joyce [Joy96b, Joy96a] as well as Kovalev [Kov03]. We refrain from reciting the history of these manifolds as it is easily found in the literature, e.g. [Joy07] has a nice summary. It is a classical fact that the restriction of a principal connection to one of its holonomy subbundles is again a connection. So, the spaces on Wang's list are those that have a G-structure for G the relevant holonomy group, in such a way that this structure inherits the Levi-Civita connection. We consider a generalisation: manifolds with a G_2 or $Spin_7$-structure to which the Levi-Civita connection does not necessarily reduce.

Manifolds with a G_2 or $Spin_7$-structure come with a rich geometry. Such a structure on a smooth manifold automatically inherits a Riemannian metric, so we do not specify one beforehand. They also come equipped with canonical differential forms; in the G_2 case a 3-form ϕ and its 4-form dual $*\phi$ and in the $Spin_7$ case a 4-form Φ. These no longer have to be parallel, however, and they are parallel if and only if the holonomy is contained within G_2 or $Spin_7$.

A U_n-structure defines a 2-form ω which, if parallel, gives us a Kähler manifold. In general ω is not parallel, and Gray and Hervella [GH80] classified U_n-structures by decomposing the covariant derivative $\nabla\omega$ into irreducibles. They did this by splitting $\mathbb{R}^{2n} \otimes \Lambda^2\mathbb{R}^{2n}$ into U_n-irreducibles, which in turn gives a splitting of the associated bundle of which $\nabla\omega$ is a section. There are four independent summands and therefore sixteen classes of U_n-structures (provided $2n > 4$; if $2n = 4$ there are four classes), many of which are well-known generalisations of Kähler manifolds. For example, one class is the nearly-Kähler spaces, another the almost-Kähler ones, another the quasi-Kähler ones, and so on. These are all special cases of the generic case, the *almost-Hermitian manifolds*.

This procedure can be carried out using G_2-structures on 7-dimensional manifolds and has been by Fernández and Gray [FG82]. For V the fundamental representation of G_2, the space $V \otimes \Lambda^3 V$ contains four independent summands. The covariant derivative $\nabla\phi$ of the canonical 3-form has four independent components, giving us sixteen classes of G_2-structure. All of these occur except one. These classes can be thought of as the analogues of the nearly-Kähler manifolds, almost-Kähler manifolds and the others. Fernández [Fer86] has used this method to classify $Spin_7$-structures, of which there are four classes. We present these classifications along with their defining differential equations in this thesis.

Similar classifications have been carried out for Sp_n-structures, also called *almost-hyper-Hermitian structures*, and for $Sp_n \cdot Sp_1$-structures, also called *almost-quaternionic-Hermitian structures*. There are at most 167 possible Sp_n-structres for $4n > 8$ (144 for $4n = 8$ and when $4n = 4$ we have $Sp_1 = SU_2$) and at most 64 $Sp_n \cdot Sp_1$-structures for $4n > 8$ (16 for $4n = 8$ and when $4n = 4$ we have $Sp_1 \cdot Sp_1 = SO_4$). These cases have been studied by Cabrera and Swann [MCS04, MC04, MCS08].

A manifold with G_2 or $Spin_7$-structure also has a naturally-defined spinor field that is parallel if and only if the holonomy is contained within G_2 or $Spin_7$. This provides a new perspective on the classification of G_2 and $Spin_7$-structures. The covariant derivative $\nabla\phi$ of the canonical 3-form can be identified with the *torsion* of the G_2-structure (see Appendix D). However, the torsion may also be identified with the covariant derivative $\nabla\varphi$ of the canonical spinor field φ. We can therefore perform an analogous procedure to the Gray-Hervella method, by decomposing the G_2-representation $V \otimes \Delta$, where Δ is the spin representation. This again gives us four independent components and therefore sixteen possible classes of G_2-structures. We derive equations relating the covariant derivatives $\nabla\phi$ and $\nabla\varphi$, for example

Theorem 1.0.5. *Let $\nabla_U \varphi = A_U \cdot \varphi$ be satisfied. If A_U is a 1-form,*

$$\nabla_U \phi(X \wedge Y \wedge Z) = -8 * \phi(A_U \wedge X \wedge Y \wedge Z) \ ,$$

i.e.

$$\nabla_U \phi = -8 \iota_{A_U} * \phi \ .$$

These comparison formulae can then be used to directly compare the two classifications of G_2-structures. We prove the expected result that they must be the same. The analogous statements hold for $Spin_7$-structures and these are included in the thesis.

The parallel differential forms on Riemannian manifolds with holonomy SU_n, Sp_n, G_2 or $Spin_7$ have several common features. One such feature is that they are all calibrations, as introduced in the work of Harvey and Lawson [HL82]. This has led many authors to try to prove differential-geometric results in G_2 and $Spin_7$ geometry analogous to those known already in Kähler geometry. Versions of deep facts like the Calabi conjecture have not yet been proven, and are known to be difficult; however, many properties concerning complex submanifolds of Kähler manifolds can be transferred. The calibrated submanifolds in G_2 geometry are the associative and coassociative submanifolds, and in $Spin_7$ geometry are the Cayley submanifolds. An excellent reference for these is Joyce's book [Joy07]. It follows immediately from the theory of calibrations that these special submanifolds are minimal. However, when we have only a G_2 or $Spin_7$-structure the canonical forms are no longer necessarily calibrations and it is not clear how to generalise that theory to prove results about associative and Cayley submanifolds in this situation. The minimality of complex submanifolds of a Kähler manifold is a classical result of Federer (see [KN69]), and does not use the properties of calibrations. Instead, we find it more useful to consider a different common property of the canonical differential forms: they are cross products.

Cross products on vector spaces were classified through algebro-topological means by Eckmann and Whitehead and by purely algebraic means by Brown and Gray (see Appendix C). Their existence on manifolds has also been studied extensively and the canonical forms on manifolds with U_n, G_2 and $Spin_7$-structures exhaust the interesting examples. The classical proof of Federer generalises easily to the cross product framework and we are able to derive a formula for the mean curvature of any associative submanifold of a manifold with a G_2-structure.

Theorem 1.0.6. *Let $\mathcal{A}$ be an associative submanifold of the manifold $\mathcal{M}$, where $\mathcal{M}$ possesses a G_2-structure. The mean curvature vector of $\mathcal{A}$ is*

$$H = -\frac{1}{12}\theta^{\perp\mathcal{A}} + 2\phi(T_{T\mathcal{A}\to N\mathcal{A}})$$

where $\theta^{\perp\mathcal{A}}$ is the component of the vector field θ orthogonal to $\mathcal{A}$ and $T_{T\mathcal{A}\to N\mathcal{A}}$ is the component of the symmetric-traceless rank two tensor field T sending $T\mathcal{A}$ to the normal bundle $N\mathcal{A}$.

We also present a formula for the mean curvature of Cayley submanifolds of manifolds with a $Spin_7$-structure. There are several natural questions to ask about this curvature and we discuss this in Chapter 11.

The third and smallest part of the thesis began as an attempt to unify the first two parts, and was motivated and supervised by José Figueroa-O'Farrill. We investigate possible analogues of Marsden-Weinstein symplectic quotients to try to product a quotient construction for manifolds with $Spin_7$ holonomy. We begin by reviewing hyper-Kähler quotients and then try to conduct a similar procedure with seven complex structures instead of three. The trouble with this is that $Spin_7$ holonomy only occurs in eight dimensions and the space Σ whose quotient we want to take needs to have higher dimension than this for the result to be non-trivial. Therefore Σ cannot be a $Spin_7$-manifold, so instead we choose Σ to be a Euclidean space with seven parallel anticommuting complex structures. There is an action of U_1 on Σ that preserves all these complex structures and acts isometrically. Level sets of the resulting moment map are U_1-invariant and their quotients $\mathcal{B}$ by the U_1-action are 8-dimensional Riemannian manifolds that inherit all seven anticommuting almost-complex structures. Using some of the basic theory of Riemannian submersions we can show that the inherited almost-complex structures are all parallel. This is at first a disappointing result because it immediately implies that our quotient manifold $\mathcal{B}$ is flat.

The required behaviour we are looking for is that the seven almost-complex structures should form a rank seven irreducible parallel subbundle of the bundle of skew-endomorphisms, and then we should have $Spin_7$ holonomy. This suggests we should look at the problem in the framework of *Clifford structures* on a Riemannian manifold, as introduced by Moroianu and Semmelmann [MS10]. They classify parallel Clifford structures and we present their tables in this thesis. We would like to take a Riemannian manifold Σ with a parallel non-flat Clifford structure and define a quotient construction whose result is a Riemannian manifold $\mathcal{B}$ with a rank seven parallel non-flat Clifford structure, which therefore must have $Spin_7$ holonomy. We have not determined whether or not this is possible; however, we can define a moment map for an isometric group action on Σ that preserves the Clifford structure. This is a moment map in the sense of Galicki and Lawson [GL88]; in fact their quaternionic-Kähler quotient moment section is a special case of ours. We attempt to prove analogues of some of the necessary propositions proved in [GL88], in order to to be able to define the *Clifford quotient* we seek. The problem is left open and we discuss this further in Chapter 11.

The structure of this thesis is summarised below:

• **Chapter** 2: The necessary definitions and basic properties of Riemannian submersions, including O'Neill's fundamental tensor fields, as well as some examples.

• **Chapter** 3: Similar definitions and facts about conformal submersions, some examples and a derivation of the relationship between the coefficients of the various Levi-Civita connections.

• **Chapter** 4: Construction of the big Clifford representation from the smaller ones, beginning with guiding examples in low dimensions and ending with isomorphisms for the general case.

• **Chapter** 5: Application of the results of the previous two chapters to a derivation of a spinorial O'Neill formula for any conformal submersion.

• **Chapter** 6: Calculation of the Dirac operators and application to the characterisation of Dirac morphisms.

• **Chapter** 7: Introduction to the background theory of G_2-structures and description of the form and spinor classifications, as well as a proof of their agreement.

• **Chapter** 8: Analogous results for $Spin_7$-structures.

• **Chapter** 9: Derivation of the mean curvature vectors of associative and Cayley submanifolds.

• **Chapter** 10: A discussion of hyper-Kähler quotients and a new '7-complex' quotient, Clifford structures and the possibility of a new construction of $Spin_7$-manifolds by a 'Clifford quotient'.

• **Chapter** 11: A look at some questions that have arisen during the creation of this thesis but so far remain unanswered.

Chapter 2

Riemannian Submersions

Riemannian submersions comprise a large and interesting class of maps between Riemannian manifolds including Riemannian products, warped products, various fibre bundles and many other non-trivial examples. In this chapter we define Riemannian submersions and describe the properties which will be useful to us later. We begin with a discussion of the basic objects involved so that we may establish notation.

2.1 Preliminaries

We identify $\mathbb{R}^p$ and $\mathbb{R}^n$ with the subspaces spanned by the first p and last n elements of the standard frame of $\mathbb{R}^{p+n}$ respectively, and correspondingly consider O_p and O_n as the subgroups

$$\begin{pmatrix} O_p & 0 \\ 0 & I_n \end{pmatrix}, \qquad \begin{pmatrix} I_p & 0 \\ 0 & O_n \end{pmatrix}$$

of O_{p+n}. We do the same for the Lie algebras of these groups $\mathfrak{so}_p$, $\mathfrak{so}_n$, yielding a splitting $\mathfrak{so}_{p+n} = \mathfrak{so}_p + \mathfrak{k} + \mathfrak{so}_n$ where $\mathfrak{k}$ is the orthogonal complement to the rest with respect to the Killing form on $\mathfrak{so}_{p+n}$. Note that $\mathfrak{k}$ is not a subalgebra in general.

Let $(\mathcal{M}, g)$ and $(\mathcal{B}, \breve{g})$ be Riemannian manifolds[1] of respective dimensions $p+n$ and n, and $\pi : \mathcal{M} \to \mathcal{B}$ a submersion, i.e. π is surjective and has surjective differential. We call $\mathcal{M}$ the *total space* and $\mathcal{B}$ the *base space* of π. The vertical bundle $\ker \pi_*$ of π is denoted by $\mathscr{V}$ and its orthogonal complement by $\mathscr{H}$, so that $T\mathcal{M} = \mathscr{V} \oplus \mathscr{H}$.

Definition 2.1.1. *A smooth map $\pi : \mathcal{M} \to \mathcal{B}$ is a* ***Riemannian submersion*** *if it is a submersion which preserves the lengths of horizontal vectors; that is, it maps $\mathscr{H}_x$ isometrically to $T_{\pi(x)}\mathcal{B}$ for every x.*

The definition of a submersion is precisely what is required for the preimage of each point b in the codomain $\mathcal{B}$ to be a submanifold of $\mathcal{M}$, which we call the *fibre* $(\mathcal{F}_b, \hat{g}_b)$ over b of π with metric $\hat{g}_b$ induced from g. The orthonormal frame bundles of $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{F}_b$ are $O(\mathcal{M})$, $O(\mathcal{B})$ and $O(\mathcal{F}_b)$, and the subbundle of $O(\mathcal{M})|_{\mathcal{F}_b}$ of frames adapted to $\mathcal{F}_b$ (those whose first p entries are tangent to $\mathcal{F}_b$) is denoted $O(\mathcal{M}, \mathscr{V})|_b$. The union over all fibres of these adapted frames is $O(\mathcal{M}, \mathscr{V})$; an $O_p \times O_n$-bundle over $\mathcal{M}$. Note that the requirement that the first p entries be vertical means that the remaining entries must be horizontal. Thus $O(\mathcal{M}, \mathscr{V})$ is precisely the subbundle of $O(\mathcal{M})$ of

[1] In this chapter we often follow the basic notation of O'Neill [O'N66] and Besse [Bes87].

frames respecting the splitting $T\mathcal{M} = \mathscr{V} \oplus \mathscr{H}$. Let $O(\mathscr{V})$ and $O(\mathscr{H})$ be the orthonormal frame bundles of the distributions $\mathscr{V}$ and $\mathscr{H}$, so that $O(\mathscr{V})|_{\mathcal{F}_b} = O(\mathcal{F}_b)$, and $O(\mathscr{H})$ is the horizontal lift of $O(\mathcal{B})$ to $\mathcal{M}$. There is a natural isomorphism[2] $O(\mathcal{M}, \mathscr{V}) \cong O(\mathscr{V}) + O(\mathscr{H})$, where the sum is used to indicate the fibrewise product as opposed to the product of the total spaces. Note that $O(\mathscr{V})$ and $O(\mathscr{H})$ are not subbundles of $O(\mathcal{M}, \mathscr{V})$, but rather quotients $O(\mathcal{M}, \mathscr{V})/O_n \cong O(\mathscr{V})$ and $O(\mathcal{M}, \mathscr{V})/O_p \cong O(\mathscr{H})$. Denote by ω, $\check{\omega}$ and $\hat{\omega}_b$ the Levi-Civita connection forms on the bundles $O(\mathcal{M})$, $O(\mathcal{B})$ and $O(\mathcal{F}_b)$, and $\hat{\omega}$ that induced on $O(\mathscr{V})$ by all the $\hat{\omega}_b$'s. Correspondingly, denote by ∇, $\check{\nabla}$ and $\hat{\nabla}$ the covariant derivative operators of these connections on the bundles $T\mathcal{M}$, $T\mathcal{B}$ and $\mathscr{V}$. The bundle $O(\mathscr{H})$ inherits a connection by pulling back $\check{\omega}$ by π_* (actually the map induced by π_* on frames). Since $O(\mathcal{M}, \mathscr{V})$ is a subbundle of $O(\mathcal{M})$ we can restrict ω to it, but this is not a connection on $O(\mathcal{M}, \mathscr{V})$ in general (it is if $\mathcal{M}$ is a Riemannian product $\mathcal{B} \times \mathcal{F}$). However, if we consider only the $\mathfrak{so}_p + \mathfrak{so}_n$ part with respect to the above splitting we do indeed get a connection on $O(\mathcal{M}, \mathscr{V})$, which we denote by $\omega^{\mathfrak{so}_p+\mathfrak{so}_n}_{O(\mathcal{M},\mathscr{V})}$. The pullbacks to $O(\mathcal{M}, \mathscr{V})$ by the homomorphisms $O(\mathcal{M}, \mathscr{V}) \to O(\mathscr{V})$ and $O(\mathcal{M}, \mathscr{V}) \to O(\mathscr{H})$ of the connections $\hat{\omega}$ and $\check{\omega}$ are the $\mathfrak{so}_p$ and $\mathfrak{so}_n$ components of $\omega^{\mathfrak{so}_p+\mathfrak{so}_n}_{O(\mathcal{M},\mathscr{V})}$:

$$\omega^{\mathfrak{so}_p+\mathfrak{so}_n}_{O(\mathcal{M},\mathscr{V})} = \begin{pmatrix} \hat{\omega} & 0 \\ 0 & \check{\omega} \end{pmatrix} ,$$

written with respect to the above decomposition of $\mathfrak{so}_{p+n}$. This fact is just Gauss' formula for a submanifold (and its dual statement) in terms of connection forms.
The map of frames induced by π, for which we use ordinary derivative notation π_*, is the map that makes

$$\begin{tikzcd}
\mathcal{M} \arrow[dd, "\pi"] & O(\mathscr{H}) \arrow[l] \arrow[dd, "\pi_*"] & \\
& & O_n \arrow[ul] \arrow[dl] \\
\mathcal{B} & O(\mathcal{B}) \arrow[l] &
\end{tikzcd}$$

commute.

Definition 2.1.2. *A vector field X on $\mathcal{M}$ is called* ***basic*** *if it is the horizontal lift by π of a vector field $\check{X}$ on $\mathcal{B}$.*

This is a very important definition, which is used repeatedly throughout this thesis.

2.2 O'Neill's tensors

Definition 2.2.1. *([O'N66]) The* ***fundamental tensor fields*** *of a Riemannian submersion are*

$$T_X Y \stackrel{\text{def}}{=} v\nabla_{vX} hY + h\nabla_{vX} vY \ ,$$

$$A_X Y \stackrel{\text{def}}{=} v\nabla_{hX} hY + h\nabla_{hX} vY$$

where ∇ is the covariant derivative of $\mathcal{M}$ and v, h are the projections onto the distributions $\mathscr{V}$, $\mathscr{H}$, and act on the vector field placed to their right.

[2] The details of all of these constructions are given in Kobayashi and Nomizu [KN69], where we have adapted them slightly to meet our needs.

While we are using the same notation v for $T\mathcal{M} \to \mathscr{V}$ and for a vertical frame $v = (v_1, \ldots, v_p)$, it will always be clear from the context which meaning is intended.

Both T and A have valency $(1,2)$, but we often do not distinguish between covariant and contravariant arguments, as is our convention. The field T is the second fundamental form of the fibres, and therefore is the obstruction to the fibres of π being totally geodesic. As explained in [O'N66], for A we have

Lemma 2.2.2. *The field A is the obstruction to the integrability of $\mathscr{H}$.*

Proof. For X, Y horizontal

$$v[X,Y] = v(\nabla_X Y - \nabla_Y X) = v(\nabla_{hX} hY - \nabla_{hY} hX) = A_X Y - A_Y X \ .$$

Now for Z a vector field on $\mathcal{B}$, and V vertical,

$$\begin{aligned}
0 &= V g(\pi^* Z, \pi^* Z) \\
&= 2g(\nabla_V \pi^* Z, \pi^* Z) \\
&= 2g(\nabla_{\pi^* Z} V + [V, \pi^* Z], \pi^* Z) \\
&= 2g(\nabla_{\pi^* Z} V, \pi^* Z) \\
&= -2g(V, \nabla_{\pi^* Z} \pi^* Z) \\
&= -2g(V, A_Z Z)
\end{aligned}$$

where we have used the fact that $[V, \pi^* Z]$ must be vertical (see [Bes87]). This means $A_Z Z$ is zero, and so

$$A_X Y = \frac{1}{2} v[X,Y] \ .$$

$\square$

Thus A is precisely the curvature in the case that $\mathscr{H}$ is an Ehresmann connection. It is interesting to note the the other two diagonal pieces of ∇, $v\nabla v$ and $h\nabla h$, are the Levi-Civita covariant derivative of the fibres $\hat{\nabla}$ and the normal covariant derivative on their normal bundles. We can add these together to get $\tilde{\nabla} = v\nabla v \oplus h\nabla h = \hat{\nabla} \oplus h\nabla h$ which is a covariant derivative on $\mathscr{V} \oplus \mathscr{H} = T\mathcal{M}$. This is precisely the covariant derivative of the connection $\omega^{\mathfrak{so}_p + \mathfrak{so}_n}_{O(\mathcal{M}, \mathscr{V})}$, and is the one used to differentiate the second fundamental form (which for a submersion is T) in Codazzi's equation for the fibres. This equation forms one of O'Neill's five fundamental equations for a Riemannian submersion, along with a Gauss equation, a pair of equations dual (i.e. corresponding to the base space) to the Gauss and Codazzi ones, and a Ricci equation. We won't list these here, but all details are in [O'N66]. The same paper also contains expressions relating the sectional curvatures of $\mathcal{M}$ and $\mathcal{B}$.

We've seen that $A_X Y$ is skew in X, Y, and similarly $T_U V$ is symmetric in U, V. It is easy to show that A_X and T_U are skew-symmetric endomorphisms. In summary, for U, V vertical and

X, Y horizontal,

$$\begin{aligned}
\nabla_U V &= \hat{\nabla}_U V + T_U V \\
\nabla_U X &= h\nabla_U X + T_U X \\
\nabla_X U &= \hat{\nabla}_X U + A_X U \\
\nabla_X Y &= h\nabla_X Y + A_X Y \ .
\end{aligned}$$

Theorem 2.2.3. *([O'N66]) Let π_1 and π_2 be Riemannian submersions $\mathcal{M} \to \mathcal{B}$. If π_1 and π_2 have the same tensors T and A and if their derivatives agree at one point of $\mathcal{M}$, then $\pi_1 = \pi_2$.*

2.3 Useful properties

Definition 2.3.1. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a submersion with vertical distribution $\mathscr{V}$. A distribution $\mathcal{H}$ which is complementary to $\mathscr{V}$, so $T\mathcal{M} = \mathscr{V} \oplus \mathcal{H}$, is called* ***complete****, or an* ***Ehresmann connection****, if for any curve $\check{\gamma}$ on $\mathcal{B}$ through b and any x in the fibre $\mathcal{F}_b$ over b there is a curve γ through x on $\mathcal{M}$ such that $\pi \circ \gamma = \check{\gamma}$ and whose tangent γ' lies in $\mathcal{H}$.*

Now assume π is Riemannian and $\mathcal{H} = \mathscr{H}$ is the horizontal distribution. Then, our distribution $\mathscr{H}$ is complete if any curve on $\mathcal{B}$ has a horizontal lift for each point in the fibre above. We state the following two facts, which can also be found in [Bes87].

Theorem 2.3.2. *([Ehr51]) If $\mathscr{H}$ is complete then π is a fibre bundle.*

Theorem 2.3.3. *([Her60]) If $(\mathcal{M}, g)$ is complete then $\mathscr{H}$ and $(\mathcal{B}, \check{g})$ are complete.*

When $\mathcal{M}$ is complete, all fibres are complete because they are closed submanifolds, and this is not hard to prove.
In the case $\mathcal{M}$ is connected and $\mathscr{H}$ is an Ehresmann connection we can speak of the typical fibre $\mathcal{F}$ and its group of symmetries $Diff(\mathcal{F})$. Note that not only does every curve through $b \in \mathcal{B}$ have for each $x \in \mathcal{F}_b$ a horizontal lift through x, but each horizontal curve through x projects nicely down to a smooth curve on $\mathcal{B}$. For an Ehresmann connection:

Definition 2.3.4. *A loop based at $b \in \mathcal{B}$ lifts to a horizontal curve beginning at any point of $\mathcal{F}_b$ and ending back on $\mathcal{F}_b$ (but in general this need not be a loop), and thus gives us a diffeomorphism of $\mathcal{F}_b$. The* ***holonomy group*** *of $\mathscr{H}$ at $b \in \mathcal{B}$, $Hol(\mathscr{H})_b$, is the subgroup of $Diff(\mathcal{F}_b)$ defined by such diffeomorphisms. Since π is locally trivial we may consider up to conjugation the holonomy group $Hol(\mathscr{H})$ as a subgroup of $Diff(\mathcal{F})$, where $\mathcal{F}$ is the typical fibre. The holonomy group is not a Lie group in general.*

Wolf proved the holonomy reduction theorem in this very general setting[3], where $\mathcal{M}$ is connected.

Theorem 2.3.5. *([Wol64]) If $\mathscr{H}$ is complete then the structure group of the fibre bundle $\mathcal{M} \to \mathcal{B}$ can be reduced to $Hol(\mathscr{H})$.*

There are many results known about Riemannian submersions with totally geodesic fibres, i.e. such that $T = 0$. The following result can be found in [Bes87].

[3] In the context of principal bundles, a reduction theorem was first published by Nomizu [Nom55]. He asserts that although unpublished until then, it was well-known at that time and was just an exact formulation of a theorem of Cartan's.

Theorem 2.3.6. *([Her60]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a Riemannian submersion with totally geodesic fibres and let $\mathscr{H}$ be complete. Then the holonomy group $Hol(\mathscr{H})_b$ is a subgroup of the isometry group of $\mathcal{F}_b$ and hence is a Lie group.*

Remark 2.3.7. *Our definition of holonomy group above uses the notion of the horizontal lift of a loop on the base to a curve on the total space. Of course, we can consider the horizontal lift of a non-closed curve on the base and in this way define* parallel transport. *Hermann [Her60] actually proved that when π has totally geodesic fibres, parallel transport is by isometries. This implies that when $\mathcal{B}$ is connected, all fibres of π are isometric. Then, we can speak of the holonomy group as a subgroup of the isometry group of the typical fibre, up to conjugation.*

Proposition 2.3.8. *([Bes87]) Let $(\mathcal{M}, g)$ be a Riemannian manifold and G a closed subgroup of the isometry group of $(\mathcal{M}, g)$. If the projection $\pi : \mathcal{M} \to \mathcal{M}/G$ is a smooth submersion then there exists a unique Riemannian metric $\breve{g}$ on $\mathcal{M}/G$ such that π is a Riemannian submersion.*

According to [Bes87], in the setting of the above proposition there always exists an open dense subset of $\mathcal{M}$ (the space of principal orbits $\mathcal{U}$ of G) such that the restriction of π is a smooth submersion. With this in mind, we have

Theorem 2.3.9. *([Bes87]) Let $(\mathcal{M}, g)$ be a complete Riemannian manifold and G a closed and connected subgroup of the isometry group of $(\mathcal{M}, g)$, such that the principal orbits of the action of G are isotropy irreducible G-homogeneous spaces. The space $\mathcal{U}$ of principal orbits is an open dense subset of $\mathcal{M}$ and $\mathcal{U} \to \mathcal{U}/G$ is a Riemannian submersion which is locally a warped product.*

We will briefly need the definition

Definition 2.3.10. *The **mean curvature vector** of $\mathscr{V}$ is*

$$\mu^{\mathscr{V}} \stackrel{\text{def}}{=} \frac{1}{p} tr^{\mathscr{V}} T = \frac{1}{p} \sum_{i=1}^{p} T_{v_i} v_i \ .$$

See Section 6.1 for a few more details.

The property of a Riemannian submersion being locally a warped product is a twisted version of a warped product. A good name for such a map is a *warped fibre bundle*. These are characterised by

Theorem 2.3.11. *([Bes87]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a warped fibre bundle. Then*

1. $A = 0$,

2. $T_U V = g(U, V)\mu^{\mathscr{V}}$,

3. $\mu^{\mathscr{V}}$ *is basic.*

Conversely, if a Riemannian submersion satisfies 1-3 then it is a warped fibre bundle.

2.4 Examples

Before we look at maps more general than Riemannian submersions, we examine some particular well-known examples of interest.

1. *Products*: The simplest kind of Riemannian submersion is a projection onto a factor of a product $\pi : \mathcal{M} = \mathcal{B} \times \mathcal{F} \to \mathcal{B}$ where $\mathcal{B}$ and $\mathcal{F}$ are Riemannian manifolds. It is clear in this case that $\mathscr{H} = \pi^* T\mathcal{B}$ is integrable and the fibres are totally geodesic, so $A = 0$ and $T = 0$. In fact the vanishing of A and T implies that $\mathcal{M}$ is *locally* a Riemannian product and π projection.

2. *Covering spaces*: A Riemannian covering space is a Riemannian submersion with $\dim \mathscr{V} = 0$, $A = 0$ and $T = 0$.

3. *Projective spaces*: The projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$ can be realised in two steps by first taking the quotient by the norm,
$$\mathbb{R}^{n+1} - \{0\} \to S^n \ , \ \mathbb{C}^{n+1} - \{0\} \to S^{2n+1} \ , \ \mathbb{H}^{n+1} - \{0\} \to S^{4n+3} \ ,$$
and then taking the quotient by the respective actions of $\mathbb{Z}_2$, U_1 and Sp_1:
$$S^n \to \mathbb{RP}^n \ , \ S^{2n+1} \to \mathbb{CP}^n \ , \ S^{4n+3} \to \mathbb{HP}^n \ .$$
One may define the usual metrics on the projective spaces to be the unique ones which make these final three maps into Riemannian submersions.

4. *Warped products*: Let $\mathcal{M} = \mathcal{B} \times_{f^2} \mathcal{F}$ and let π be projection onto the *first* factor. Then π is a Riemannian submersion and 2.3.11 applies. In general it is not true that $T = 0$ so Remark 2.3.7 does not apply and the fibres need not be isometric. They are homothetic to one another with factor f^2, however.

5. *Tangent bundles*: The tangent bundle $\pi : T\mathcal{B} \to \mathcal{B}$ of a Riemannian manifold where $T\mathcal{B}$ is equipped with the Sasaki metric (as originally defined by Sasaki in [Sas58]) is a Riemannian submersion. The tensor field A is precisely the curvature 2-form with values in the vertical bundle of the linear connection on $T\mathcal{B}$ (the Ehresmann connection transferred to $T\mathcal{B}$ from the Levi-Civita connection on $O(\mathcal{B})$ through the associated bundle construction). The fibres are totally geodesic so $T = 0$ and by Remark 2.3.7 parallel transport must be by isometries. Of course, this is a classical fact of Riemannian geometry.

6. *Orthonormal frame bundles*: The orthonormal frame bundle $\pi : O(\mathcal{B}) \to \mathcal{B}$ of a Riemannian manifold where $O(\mathcal{B})$ is equipped with the Sasaki-Mök metric (as originally defined by O'Neill [O'N66] and later by Mök [Mök78], apparently without reference to O'Neill's paper[4]) is a Riemannian submersion. The tensor field A is precisely the curvature 2-form of the Levi-Civita connection. The fibres are totally geodesic so $T = 0$ here too.

7. *More orthonormal frame bundles*: As we have seen, a Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$ induces an O_n-bundle map $O(\mathscr{H}) \to O(\mathcal{B})$. It is not difficult to show that when $O(\mathscr{H})$

[4] This resulted in more published papers without reference to O'Neill. This metric and related ones have been studied for example in [KS08a], [KS08b] and [Sek08]

and $O(\mathcal{B})$ are equipped with their Sasaki-Mök metrics this map $O(\mathscr{H}) \to O(\mathcal{B})$ is a Riemannian submersion. In this way we get a commuting diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[d, "\pi"] & O(\mathscr{H}) \arrow[l] \arrow[d, "\pi_*"] & O(\mathscr{H}_2) \arrow[l] \arrow[d, "\pi_{**}"] & O(\mathscr{H}_3) \arrow[l] \arrow[d, "\pi_{***}"] & \cdots O(\mathscr{H}_k) \arrow[d, "\pi_{*k}"] \\
\mathcal{B} & O(\mathcal{B}) \arrow[l] & O(O(\mathcal{B})) \arrow[l] & O(O(O(\mathcal{B}))) \arrow[l] & \cdots O^k(\mathcal{B})
\end{tikzcd}$$

where every arrow is a Riemannian submersion.

Chapter 3

Conformal Submersions

Now the scene for Riemannian submersions has been set it does not require a great leap of imagination to generalise to submersions that are only conformal. Here we explain the important differences and then calculate relations between the connection coefficients which we will need in a later chapter.

3.1 Preliminaries

Suppose now that our submersion is not Riemannian, but instead satisfies only a weaker property.

Definition 3.1.1. *A* ***conformal submersion*** $\pi : \mathcal{M} \to \mathcal{B}$ *between Riemannian manifolds is a submersion which preserves angles between horizontal vectors.*

In this case the derivative π_* at $x \in \mathcal{M}$ doesn't map $\mathscr{H}_x$ isometrically to $T_{\pi(x)}\mathcal{B}$, but only conformally. Then, for each x we get a number by which elements in $\mathscr{H}_x$ increase in length, i.e. a function on $\mathcal{M}$. It is conventional to instead use the square reciprocal of this, that is, the amount by which the metric is scaled up. This function is called the *conformal factor* of the submersion, and will be usually be denoted by λ^2—in the form of a square because we will mostly be writing its square root.

Remark 3.1.2. *The foundational papers [Fug78] and [Ish79] also use the term 'conformal submersion' for such maps, however in some papers extra words seem to have crept in. In [OR93], these maps are called 'horizontally weakly conformal'.*

Most of the comments made in 2.1 remain valid—we can define $O(\mathscr{V})$, $O(\mathscr{H})$ and $O(\mathcal{M}, \mathscr{V})$ in the same way. The important difference is that π_* no longer sends $O(\mathscr{H})$ to $O(\mathcal{B})$. A horizontal orthonormal frame on $\mathcal{M}$ is of the form $(\lambda^{-1}e_1, \ldots, \lambda^{-1}e_n)$ where $(e_1, \ldots, e_n)$ is orthonormal on $\mathcal{B}$, where we use the same notation e_i for the vector field on $\mathcal{B}$ and its pullback to $\mathcal{M}$ by π. So instead of π_*, we must use

$$\Xi \stackrel{\text{def}}{=} [\pi_* \lambda] : O(\mathscr{H}) \to O(\mathcal{B})$$

which means: first multiply by λ and then apply π_*. The diagram

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\pi"] & O(\mathscr{H}) \arrow[l] \arrow[d, "\Xi"] & \\ & & O_n \arrow[ul] \arrow[dl] \\ \mathcal{B} & O(\mathcal{B}) \arrow[l] & \end{tikzcd}$$

commutes. As in the Riemannian case, the bundle $O(\mathscr{H})$ inherits a connection by pulling back the Levi-Civita connection $\check{\omega}$ of $O(\mathcal{B})$ using Ξ; the 1-form $\Xi^*\check{\omega}$ is $\mathfrak{so}_n$-valued and equivariant and reproduces the vertical Lie algebra homomorphisms since $\Xi_*\iota_q = \iota_{\Xi q}$ where ι_q is the canonical identification map $\mathfrak{so}_n \to \mathscr{V}_q$. The connection $\hat{\omega}$ on $O(\mathscr{V})$ does not depend on whether the submersion π is Riemannian or conformal, and so is as before. The difference from the case of Riemannian submersions is that the pullback to $O(\mathcal{M}, \mathscr{V})$ of $\Xi^*\check{\omega}$ is not in general equal to the $\mathfrak{so}_n$ component of the restriction $\omega_{O(\mathcal{M},\mathscr{V})}$.

Definition 3.1.3. *([OR93]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion with conformal factor λ, where $\mathcal{M}$ and $\mathcal{B}$ are equipped with the metrics g and $\check{g}$ respectively. The **associated Riemannian submersion** is given by the same map of smooth manifolds π, but where $\mathcal{M}$ is given the metric $\lambda^{-2}g$ instead of g.*

We can derive the relationships between the fundamental tensor fields T^g and A^g of a conformal submersion and those of its associated Riemannian submersion $T^{\lambda^{-2}g}$ and $A^{\lambda^{-2}g}$. Using the well-known formula for a conformal change of $\lambda^{-2}g$ to g

$$\nabla^g_X Y = \nabla^{\lambda^{-2}g}_X Y + \frac{1}{\lambda}\left[X(\lambda)Y + Y(\lambda)X - g(X,Y)grad^g\lambda\right] ,$$

it is easy to show that

$$A^g_X = A^{\lambda^{-2}g}_X - \frac{2}{\lambda}X \wedge vgrad^g\lambda$$

for any X. A similar relationship holds between T^g_U and $T^{\lambda^{-2}g}_U$. We know that the tensor field $A^{\lambda^{-2}g}$ is the obstruction to the integrability of the horizontal distribution (which remains unchanged under conformal rescaling), but the new version A^g is not. The proof of this property fails for A^g because unlike $A^{\lambda^{-2}g}$ it is not antisymmetric when acting on horizontal vector fields. The new field T^g is the obstruction to the fibres being totally umbilic submanifolds.

Using 2.2.3 it is clear that T^g and A^g determine the conformal submersion up to conformal diffeomorphisms, provided we are given the original metric on $\mathcal{M}$ and hence, the conformal factor. This appears in [OR93], along with conformal analogues of the Gauss-Codazzi-Ricci equations for Riemannian submersions proved by O'Neill.

3.2 Examples

1. *Riemannian cones*: For $\mathcal{B}$ a Riemannian manifold the warped product $\mathcal{M} = \mathbb{R} \times_{f^2} \mathcal{B}$, where $f : \mathbb{R} \to \mathbb{R}$ is the identity $f(r) = r$, is called the Riemannian cone over $\mathcal{B}$. Projection onto $\mathcal{B}$ is a conformal submersion with factor r^2. The covariant derivatives of $\mathcal{M}$ and $\mathcal{B}$

are simply related by the well-known equations (see for instance [Bär93])

$$\nabla_X Y = \check{\nabla}_X Y - \frac{1}{r} g(X,Y)\partial_r \ ,$$

$$\nabla_{\partial_r} X = \nabla_X \partial_r = \frac{1}{r} X \ , \ \nabla_{\partial_r}\partial_r = 0$$

for X and Y vector fields on $\mathcal{B}$ and ∂_r the unique unit oriented vector field on $\mathbb{R}$. From these we can see that for Riemannian cones $T^g = 0$ and

$$A^g_X = -\frac{2}{r} X \wedge \partial_r \ .$$

The associated Riemannian submersion is projection onto $\mathcal{B}$ from the cylinder $\mathbb{R} \times \mathcal{B}$. To see this, denote the cone metric by $g = dr \otimes dr + r^2 \check{g}$ so that the metric of the associated Riemannian submersion is $dr \otimes dr / r^2 + \check{g}$. A change of coordinate to $t = \log r$ does the trick. From the relationship between A^g and $A^{\lambda^{-2}g}$ we must have

$$A^{\lambda^{-2}g} = 0$$

as we expect, since $\mathbb{R} \times \mathcal{B}$ is just a Riemannian product.

2. *Warped products*: Let $\mathcal{M} = \mathcal{B} \times_{f^2} \mathcal{F}$ and let π be projection onto the *second* factor. Then π is a conformal submersion with conformal factor f^2. The warped product metric can be written $g = \check{g} + f^2 \hat{g}$, and then the metric of the associated Riemannian submersion is $\check{g}/f^2 + \hat{g}$. This is not isometric to the ordinary Riemannian product unless either $f^2 = 1$ or the dimension of $\mathcal{B}$ is 1, as for the cone above.

3. *Projective spaces*: The maps

$$\mathbb{R}^{n+1} - \{0\} \to S^n \ , \ \mathbb{C}^{n+1} - \{0\} \to S^{2n+1} \ , \ \mathbb{H}^{n+1} - \{0\} \to S^{4n+3}$$

are conformal submersions with conformal factors given by the square norm on each of $\mathbb{R}^{n+1}$, $\mathbb{C}^{n+1}$ and $\mathbb{H}^{n+1}$. By composing with the maps $S^n \to \mathbb{RP}^n$, $S^{2n+1} \to \mathbb{CP}^n$ and $S^{4n+3} \to \mathbb{HP}^n$ we obtain conformal submersions

$$\mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n \ , \ \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n \ , \ \mathbb{H}^{n+1} - \{0\} \to \mathbb{HP}^n$$

whose conformal factors are again given by the square norm.

4. *Tangent bundles*: The Sasaki metric on the total space of the tangent bundle of a Riemannian manifold can be modified as follows. Let $TT\mathcal{B} = \mathscr{V} \oplus \mathscr{H}$ be the splitting into the vertical distribution and the Levi-Civita connection, and let $g = \hat{g} + \check{g}$ be the Sasaki metric written with respect to this splitting (as it is defined—see [O'N66]). We define $g_{\lambda^2} = \hat{g} + \lambda^2 \check{g}$ on $TT\mathcal{B}$ and call it a *skewed Sasaki metric*. Then projection $T\mathcal{B} \to \mathcal{B}$ is a conformal submersion with conformal factor λ^2. Such metrics have been studied in [WW11].

5. *Orthonormal frame bundles*: It is also possible to skew the Sasaki-Mök metric on $O(\mathcal{B})$ in the same way to get a conformal submersion $O(\mathcal{B}) \to \mathcal{B}$. Such metrics have been studied in [KS08b] and [Sek08]. It is also possible to skew the metric in the vertical direction to

obtain even more conformal submersions, and this does not change the conformal factor.

6. *More orthonormal frame bundles*: It should be noted that $\Xi : O(\mathscr{H}) \to O(\mathcal{B})$ is never a conformal submersion unless it is Riemannian, which happens when $\pi : \mathcal{M} \to \mathcal{B}$ is Riemannian as in 7 of 2.4.

3.3 Connection coefficients

In this section we shall derive expressions for the coefficients of the Levi-Civita connection of the total space $\mathcal{M}$ of a conformal submersion in terms of the coefficients of the Levi-Civita connections of the base space and fibres.

Definition 3.3.1. *The* ***horizontal covariant derivative*** $\nabla^{\mathscr{H}}$ *is the covariant derivative operator on* $\mathscr{H}$ *given by* $\Xi^*\check{\omega}$.

Proposition 3.3.2. *Let* $\lambda^{-1}e$ *be a local orthonormal frame of* $\mathscr{H}$*, so* $\Xi(\lambda^{-1}e) = e$ *is orthonormal on* $\mathcal{B}$.

1. *The coefficients of* $\nabla^{\mathscr{H}}$ *and* $\pi^*\check{\nabla}$ *satisfy*

$$\Gamma^{\lambda^{-1}e}_{\mathscr{H}} = \pi^*\check{\Gamma}^e \ ,$$

2. *For* X *basic and* $d^{\lambda^{-1}e}$ *the derivative part with respect to the coordinates relative to* $\lambda^{-1}e$,

$$d^{\lambda^{-1}e}X = \frac{d\lambda}{\lambda} \otimes X + (\pi^* d^e)X \ ,$$

3. *Thus*

$$\nabla^{\mathscr{H}} X = \frac{d\lambda}{\lambda} \otimes X + (\pi^*\check{\nabla})X \ .$$

Proof. If

$$\check{X} = \sum_{i=1}^{n} \check{X}^i e_i$$

then the basic lift X is

$$X = \sum_{i=1}^{n} (\check{X}^i \circ \pi) e_i = \sum_{i=1}^{n} \lambda(\check{X}^i \circ \pi)(\lambda^{-1}e_i)$$

where we use the same symbol for the pullback of e by π and e itself. Using the associated bundle notation for $T\mathcal{B} = O(\mathcal{B}) \times_{O_n} \mathbb{R}^n$ and $\mathscr{H} = O(\mathscr{H}) \times_{O_n} \mathbb{R}^n$ we have

$$\check{X} = [e, \check{X}_0] \ ,$$

where $\check{X}_0$ is a locally-defined $\mathbb{R}^n$-valued function on $\mathcal{B}$. The basic lift of $\check{X}$ is thus

$$X = [\lambda^{-1}e, \lambda\check{X}_0 \circ \pi] \ .$$

To prove *1*, we note

$$\Gamma^{\lambda^{-1}e}_{\mathscr{H}} = (\lambda^{-1}e)^*\Xi^*\check{\omega} = \check{\omega}(\Xi_*(\lambda^{-1}e)_*\cdot)$$

and $\Xi_*(\lambda^{-1}e)_* = \big[\Xi(\lambda^{-1}e)\big]_* = e_*$. Next

$$\begin{aligned}(\pi^* d^e)X &= \pi^*(d^e \check{X})\\ &= \pi^*\big(d^e[e,\check{X}_0]\big)\\ &= \pi^*[e, d\check{X}_0]\\ &= [\lambda^{-1}e, \lambda d\check{X}_0 \circ \pi_*]\end{aligned}$$

and

$$\begin{aligned}d^{\lambda^{-1}e}X &= d^{\lambda^{-1}e}[\lambda^{-1}e, \lambda\check{X}_0 \circ \pi]\\ &= d\lambda \otimes [\lambda^{-1}e, \check{X}_0 \circ \pi] + [\lambda^{-1}e, \lambda d\check{X}_0 \circ \pi_*]\\ &= \frac{d\lambda}{\lambda} \otimes X + [\lambda^{-1}e, \lambda d\check{X}_0 \circ \pi_*]\end{aligned}$$

which proves *2*. We can prove *3* either in the same notation by writing

$$(\pi^*\check{\nabla})X = [\lambda^{-1}e, \lambda d\check{X}_0 \circ \pi_* + \lambda\check{\omega}(e_*\cdot)\check{X}_0 \circ \pi]$$

and

$$\begin{aligned}\nabla^{\mathscr{H}}X &= \nabla^{\mathscr{H}}[\lambda^{-1}e, \lambda\check{X}_0 \circ \pi]\\ &= \frac{d\lambda}{\lambda} \otimes X + [\lambda^{-1}e, \lambda d\check{X}_0 \circ \pi_* + \lambda\check{\omega}(\Xi_*(\lambda^{-1}e)_*\cdot)\check{X}_0 \circ \pi]\ ,\end{aligned}$$

or we could equivalently write $\check{\nabla}_{\check{X}}\check{Y} = d^e\check{Y}(\check{X}) + \check{\Gamma}^e_{\check{X}}\check{Y}$, or

$$(\pi^*\check{\nabla})_X Y = (\pi^* d^e)Y(X) + (\pi^*\check{\Gamma}^e)_X Y$$

and

$$\nabla^{\mathscr{H}}_X Y = d^{\lambda^{-1}e}Y(X) + \Gamma^{\lambda^{-1}e}_{\mathscr{H}X}Y\ ,$$

from which the result follows from the first two parts. □

Let $(v, \lambda^{-1}e)$ be a local section of $O(\mathcal{M}, \mathscr{V})$. Then for the covariant derivatives on the bundles $\mathscr{V}$, $\mathscr{H}$ and $T\mathcal{M}$ we can write

$$\begin{aligned}\hat{\nabla} &= d^v + \hat{\Gamma}^v\ ,\\ \nabla^{\mathscr{H}} &= d^{\lambda^{-1}e} + \Gamma^{\lambda^{-1}e}_{\mathscr{H}}\ ,\\ \nabla &= d^{(v,\lambda^{-1}e)} + \Gamma^{(v,\lambda^{-1}e)}\end{aligned}$$

where the superscripts refer to the local sections of the appropriate frame bundles used. It is easy to see that for $Y \in \mathfrak{X}(\mathcal{M})$

$$d^{(v,\lambda^{-1}e)}Y = d^v vY + d^{\lambda^{-1}e}hY\ ,$$

i.e. d^v and $d^{\lambda^{-1}e}$ are the vertical and horizontal parts of $d^{(v,\lambda^{-1}e)}$.
Our purpose in this section is to write expressions for $\Gamma^{(v,\lambda^{-1}e)}_X$ and $\Gamma^{(v,\lambda^{-1}e)}_U$, where X is horizontal and U is vertical, in terms of the connection coefficients of $\check{\omega}$ and $\hat{\omega}$, and the conformal

factor λ^2. We will also state which terms correspond to the summands in the splitting $\mathfrak{so}_{p+n} = \mathfrak{so}_p + \mathfrak{k} + \mathfrak{so}_n$.

Proposition 3.3.3. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian manifolds. The coefficients $\Gamma^{(v,\lambda^{-1}e)}$ of the Levi-Civita connection ω with respect to the local adapted frame $(v, \lambda^{-1}e)$ satisfy*

1. $$\Gamma_X^{(v,\lambda^{-1}e)} = \pi^*\check{\Gamma}_X^e - \frac{2}{\lambda} X \wedge hgrad^g\lambda + A_X^g + \hat{\Gamma}_X^{(v,\lambda^{-1}e)}$$

   when X is horizontal, where the $\mathfrak{so}_n$-part is $\pi^*\check{\Gamma}_X^e - \frac{2}{\lambda} X \wedge hgrad^g\lambda$, the $\mathfrak{k}$-part is A_X^g and the $\mathfrak{so}_p$-part is $\hat{\Gamma}_X^{(v,\lambda^{-1}e)}$,

2. $$\Gamma_U^{(v,\lambda^{-1}e)} = A^g U + T_U^g + \hat{\Gamma}_U^{(v,\lambda^{-1}e)}$$

   when U is vertical, where the $\mathfrak{so}_n$-part is $A^g U$, the $\mathfrak{k}$-part is T_U^g and the $\mathfrak{so}_p$-part is $\hat{\Gamma}_U^{(v,\lambda^{-1}e)}$.

Proof. The coefficients $\Gamma^{(v,\lambda^{-1}e)}$ are the pullback of the connection ω using the local section $(v, \lambda^{-1}e)$ of $O(\mathcal{M}, \mathscr{V}) \subset O(\mathcal{M})$, and form a 1-form on a patch of $\mathcal{M}$ with values in the skew endomorphisms. The $\mathfrak{so}_p$, $\mathfrak{k}$, $\mathfrak{so}_n$ parts of ω then correspond to the parts of $\Gamma^{(v,\lambda^{-1}e)}$ sending $\mathscr{V}$ to $\mathscr{V}$, swapping $\mathscr{H}$ and $\mathscr{V}$ and sending $\mathscr{H}$ to $\mathscr{H}$ respectively. We begin by finding $\Gamma_X^{(v,\lambda^{-1}e)}$. Let X, Y, Z be the basic lifts of $\check{X}$, $\check{Y}$, $\check{Z}$. From the Koszul formula

$$\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
&\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \\
&= 2\lambda\big[X(\lambda)\check{g}(\check{Y},\check{Z}) + Y(\lambda)\check{g}(\check{Z},\check{X}) - Z(\lambda)\check{g}(\check{X},\check{Y})\big] \\
&\quad + \lambda^2\big[X\check{g}(\check{Y},\check{Z}) + Y\check{g}(\check{Z},\check{X}) - Z\check{g}(\check{X},\check{Y}) \\
&\qquad - \check{g}(\check{X},[\check{Y},\check{Z}]) + \check{g}(\check{Y},[\check{Z},\check{X}]) + \check{g}(\check{Z},[\check{X},\check{Y}])\big] \\
&= 2\lambda\big[X(\lambda)\check{g}(\check{Y},\check{Z}) + Y(\lambda)\check{g}(\check{Z},\check{X}) - Z(\lambda)\check{g}(\check{X},\check{Y})\big] \\
&\qquad + 2\lambda^2\check{g}(\check{\nabla}_{\check{X}}\check{Y},\check{Z}) \ .
\end{aligned}$$

Note that since Z is horizontal we know $g(\nabla_X Y, Z) = g(h\nabla_X Y, Z)$, and we also have $g(h\nabla_X Y, Z) = \lambda^2\check{g}(h\nabla_X Y, \check{Z})$, and $Z(\lambda) = g(grad^g\lambda, Z) = \lambda^2\check{g}(hgrad^g\lambda, \check{Z})$. Thus

$$h\nabla_X Y = \check{\nabla}_{\check{X}}\check{Y} + \frac{1}{\lambda}\big[X(\lambda)Y + Y(\lambda)X \ - g(X,Y)hgrad^g\lambda\big] \ .$$

This is our generalisation of the well-known formula for conformal changes of metric presented in 3.1. The vertical part of $\nabla_X Y$ is $A_X^g Y$, so in total we have

$$\nabla_X Y = \check{\nabla}_{\check{X}}\check{Y} + \frac{1}{\lambda}\big[X(\lambda)Y + Y(\lambda)X - g(X,Y)hgrad^g\lambda\big] + A_X^g Y \ .$$

Proposition 3.3.2 then gives

$$\nabla_X Y = \nabla_X^{\mathscr{H}} Y + \frac{1}{\lambda}\big[Y(\lambda)X - g(X,Y)hgrad^g\lambda\big] + A_X^g Y$$

from which we see that the part of the coefficient sending $\mathscr{H}$ to $\mathscr{H}$ is

$$\pi^*\check{\Gamma}^e_X + \frac{1}{\lambda}\big[d\lambda(h\cdot)\otimes X - g(X,\cdot)hgrad^g\lambda\big] = \pi^*\check{\Gamma}^e_X - \frac{2}{\lambda}X\wedge hgrad^g\lambda\ .$$

The part sending $\mathscr{H}$ to $\mathscr{V}$ is $A^g_X \circ h$. We see no $\mathfrak{so}_p$ part in this expression because Y is horizontal. Also

$$\nabla_X U = h\nabla_{hX}vU + v\nabla_{hX}vU = A^g_X U + \hat{\nabla}_X U$$

tells us that the component sending $\mathscr{V}$ to $\mathscr{H}$ is $A^g_X \circ v$ (so the total $\mathfrak{k}$-part is A^g_X) and the $\mathfrak{so}_p$-part here is as claimed.
To find the $\mathfrak{so}_n$-part of $\Gamma^{(v,\lambda^{-1}e)}_U$:

$$\begin{aligned} h\nabla_U X &= h\big(\nabla_X U + [U,X]\big) \\ &= h\big(h\nabla_{hX}vU + v\nabla_{hX}vU + [U,X]\big) \\ &= A^g_X U + h[U,X]\ . \end{aligned}$$

On the other hand $h\nabla_U X = hd^{(v,\lambda^{-1}e)}X(U) + h\Gamma^{(v,\lambda^{-1}e)}_U X$. If X is basic

$$d^{(v,\lambda^{-1}e)}X(U) = \sum_{i=1}^{n} dX^i \otimes (\lambda^{-1}e_i)(U) = 0\ .$$

Observing that when U is vertical and X basic, the field $[U,X]$ is vertical, we find that

$$h\Gamma^{(v,\lambda^{-1}e)}_U X = A^g_X U$$

when X is basic. But this is a tensorial expression, so must hold for any horizontal X. Note also that

$$v\nabla_U X = v\nabla_{vU}hX = T^g_U X$$

so the component of $\Gamma^{(v,\lambda^{-1}e)}_U$ sending $\mathscr{H}$ to $\mathscr{V}$ is $T^g_U \circ h$. Finally for V vertical

$$\nabla_U V = T^g_U V + \hat{\nabla}_U V$$

which means the component sending $\mathscr{V}$ to $\mathscr{H}$ is $T^g_U \circ v$ (so the total $\mathfrak{k}$-part is T^g_U) and the $\mathfrak{so}_p$-part is $\hat{\Gamma}^{(v,\lambda^{-1}e)}_U$, which completes the proof. □

3.4 Homothetic submersions

As we have already remarked in 3.1, for a conformal submersion the pullback to $O(\mathcal{M},\mathscr{V})$ of $\Xi^*\check{\omega}$ is not in general equal to the $\mathfrak{so}_n$ component[1] of the restriction $\omega_{O(\mathcal{M},\mathscr{V})}$ of ω to $O(\mathcal{M},\mathscr{V}) \subset O(\mathcal{M})$. Is it ever equal and if so, when?
The answer to this question is immediate from Proposition 3.3.3—the the pullback to $O(\mathcal{M},\mathscr{V})$ of $\Xi^*\check{\omega}$ is equal to the $\mathfrak{so}_n$ component of the restriction $\omega_{O(\mathcal{M},\mathscr{V})}$ exactly when $hgrad^g\lambda = 0$. This condition defines a subclass of conformal submersions which share several of the properties of Riemannian submersions not shared by the others. As mentioned earlier, the horizontal

[1] Gauss' formula tells us that the pullback of $\hat{\omega}$ on $O(\mathscr{V})$ via the quotient $O(\mathcal{M},\mathscr{V}) \to O(\mathscr{V})$ corresponds to the $\mathfrak{so}_p$ part of $\omega_{O(\mathcal{M},\mathscr{V})}$—this remains true in the conformal case because the fibres are Riemannian submanifolds of $\mathcal{M}$ and their inclusions make no mention of any geometric properties of π.

distribution $\mathscr{H}$ generalizes a connection (see Theorem 2.3.2). In this spirit, we make the following definition.

Definition 3.4.1. *The* ***absolute differential*** *of a form on the total space $\mathcal{M}$ of a conformal submersion is $D^\pi = h^*d$, i.e. it is the horizontal partial derivative.*

It is clear that the condition $D^\pi\lambda = 0$ is equivalent to the conformal factor λ^2 being constant along any horizontal curve, i.e. $hgrad^g\lambda = 0$.

Definition 3.4.2. *Let π be a conformal submersion of Riemannian manifolds. Then we say π is* ***homothetic*** *if its conformal factor λ^2 satisfies $D^\pi\lambda = 0$.*

We have proved

Theorem 3.4.3. *The pullback of $\Xi^*\breve{\omega}$ to $O(\mathcal{M}, \mathscr{V})$ agrees with the $\mathfrak{so}_n$ part of $\omega_{O(\mathcal{M},\mathscr{V})}$ if and only if π is homothetic.*

The use of the term 'homothetic submersion' was motivated by what is now a corollary for the case when π has no kernel: a conformal diffeomorphism of Riemannian manifolds is connection-preserving[2] if and only if it is a homothety in the usual sense.

Observe that the warping factor f^2 of a warped product $\mathcal{M} = \mathcal{B} \times_{f^2} \mathcal{F}$ depends only on the point in $\mathcal{B}$. This warping factor is also the conformal factor of the second projection as a submersion. A horizontal curve of this submersion is a curve in $\mathcal{F}$ with fixed $\mathcal{B}$-coordinate, and so the conformal factor does not vary along horizontal curves. The second projection of any warped product is therefore homothetic, including the projection of a Riemannian cone as described in 3.2.

[2]The term *connectomorphism* is gaining currency.

Chapter 4

The Spin Space of a Sum

This chapter contains the algebra necessary to understand the spin space associated to an orthogonal sum of inner product spaces in terms of the spin spaces associated to the summands. We will focus our attention on the case when the summands each have dimension greater than one, as the situation where a summand is one-dimensional is well-known and can be found throughout the literature. We expect exponential behaviour, and indeed we shall find that, roughly speaking, the taking of spin spaces changes a direct sum into a tensor product. However, the precise scenarios are quite complicated and must be treated in several different ways: the case where both dimensions are even, when one is even and the other is odd, and when both are odd. The case where both dimensions are even is understood[1] and is used, for example, in [LS09]. We will present the findings in the same way they were researched, by looking at low-dimensional examples in detail and using these as guidance to form general statements.

4.1 Two plus two dimensions

Consider the vector space $\mathbb{R}^2$ with its standard positive-definite inner product. The complex Clifford algebra of $\mathbb{R}^2$ is denoted $\mathbb{C}l_2$ and is non-canonically isomorphic to the complex algebra of 2×2 complex matrices $\mathbb{C}(2)$. Various isomorphisms $\mathbb{C}l_2 \xrightarrow{\sim} \mathbb{C}(2)$ can be chosen so that the involution appears as

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} .$$

To explain what we mean by various: one can check that the group of complex automorphisms of $\mathbb{C}(2)$ which commute with the involution as written here is isomorphic to the two-component universal covering group of the second orthogonal group O_2.

A choice of isomorphism $\mathbb{C}l_2 \xrightarrow{\sim} \mathbb{C}(2)$ is equivalent to a choice of frame for the unique irreducible complex representation Δ_2 of $\mathbb{C}l_2$ which is adapted to the splitting

$$\Delta_2^0 \stackrel{\text{def}}{=} \Delta_2|_{\mathbb{C}l_2^0} = \Delta_2^{0+} \oplus \Delta_2^{0-}$$

into one-dimensional subrepresentations (semispinors), where the even part $\mathbb{C}l_2^0$ of $\mathbb{C}l_2$ is non-canonically isomorphic to $\mathbb{C} \oplus \mathbb{C}$. The Clifford representation Δ_2 is thus identified with $\mathbb{C}^2$ and the action given by ordinary matrix multiplication.

[1]Unbeknownst to the author at the time of this research, a different construction achieving the same goal in the harder cases is described in [Bär98].

If we add two copies of $\mathbb{R}^2$ together, what happens to the Clifford representations?

It is a well-known fact that

$$\mathbb{C}l_2 \mathbin{\hat{\otimes}} \mathbb{C}l_2 \overset{\text{canon}}{\cong} \mathbb{C}l_4$$

where $\hat{\otimes}$ is $\mathbb{Z}_2$-graded tensor product. The two copies of $\mathbb{C}l_2$ are realised naturally as subalgebras by $\mathbb{C}l_2 \to \mathbb{C}l_2 \mathbin{\hat{\otimes}} 1$ and $\mathbb{C}l_2 \to 1 \mathbin{\hat{\otimes}} \mathbb{C}l_2$. Recall that $\mathbb{C}l_4$ is non-canonically isomorphic to $\mathbb{C}(4)$. This means that whilst the above isomorphism is natural, an isomorphism $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$ is not.

The product on $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2)$, where the $\mathbb{Z}_2$-gradings are induced in the above way from $\mathbb{C}l_2$, is given by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \cdot \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} \hat{\otimes} \begin{pmatrix} E & F \\ G & H \end{pmatrix}\right]$$

$$\begin{aligned}
= & \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \\
& \cdot \left[\begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \begin{pmatrix} E & F \\ G & H \end{pmatrix} + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \begin{pmatrix} E & F \\ G & H \end{pmatrix}\right]
\end{aligned}$$

$$\begin{aligned}
= & \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} \\
& + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} \\
& + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} \\
& - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix}.
\end{aligned}$$

The minus sign in the final term is what makes this a $\mathbb{Z}_2$-graded algebra; it is present when the two inner 2×2 matrices are odd with respect to the involution on $\mathbb{C}(2)$.

One of many isomorphisms $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} Aa & Bb & Ba & Ab \\ -Cc & Dd & -Dc & Cd \\ Ca & -Db & Da & -Cb \\ Ac & Bd & Bc & Ad \end{pmatrix}.$$

This was found by first finding an arbitrary isomorphism $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$, then composing with a couple of automorphisms of $\mathbb{C}(4)$ to get the above more useful form. Without the minus signs the map gives an isomorphism $\mathbb{C}(2) \otimes \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$. Since $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2)$ inherits its involution from the two copies of $\mathbb{C}(2)$, our map induces an involution on $\mathbb{C}(4)$. The first isomorphism $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$ we found induced an involution given by an inconvenient array of minus signs; our map above induces the involution given by putting minus signs in positions

$$\begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{pmatrix}.$$

This is more convenient because the natural complex representation $\mathbb{C}^4$ of $\mathbb{C}(4)$ then inherits an involution

$$\overline{\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

which means that we can treat elements as pairs of two-dimensional vectors and repeat what we have to say about Δ_2 for Δ_4. This will be more apparent later.

Since $\mathbb{C}(2) \hat{\otimes} \mathbb{C}(2) \cong \mathbb{C}(4)$, the algebra $\mathbb{C}(2) \hat{\otimes} \mathbb{C}(2)$ must have a unique irreducible complex representation. Since Δ_2 possesses an involution, given by

$$\overline{\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}} = \begin{pmatrix} \varphi_1 \\ -\varphi_2 \end{pmatrix}$$

with respect to our choice of isomorphism $\mathbb{C}l_2 \xrightarrow{\sim} \mathbb{C}(2)$, we can define an action of $\mathbb{C}(2) \hat{\otimes} \mathbb{C}(2)$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by

Definition 4.1.1. *The $\mathbb{Z}_2$-graded complex representation $\mathbb{C}^2 \hat{\otimes} \mathbb{C}^2$ of $\mathbb{C}(2) \hat{\otimes} \mathbb{C}(2)$ acting on the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \otimes \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

$$\stackrel{\text{def}}{=} \begin{pmatrix} a\varphi_1 + b\varphi_2 \\ c\varphi_1 + d\varphi_2 \end{pmatrix} \otimes \begin{pmatrix} A\chi_1 \\ D\chi_2 \end{pmatrix} + \begin{pmatrix} a\varphi_1 - b\varphi_2 \\ c\varphi_1 - d\varphi_2 \end{pmatrix} \otimes \begin{pmatrix} B\chi_2 \\ C\chi_1 \end{pmatrix}.$$

We also use $\hat{\otimes}$ for the elements of $\mathbb{C}^2 \hat{\otimes} \mathbb{C}^2$.

Remark 4.1.2. *The use of the symbol $\hat{\otimes}$ is due to the fact that this is a $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded representations. We will abuse this notation in the other cases—see Remarks 4.2.2 and 4.3.2.*

A calculation shows that this is indeed an irreducible representation.

Proposition 4.1.3. *The isomorphism $\mathbb{C}^2 \hat{\otimes} \mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^4$*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \hat{\otimes} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \to \begin{pmatrix} \varphi_1\chi_1 \\ -\varphi_2\chi_2 \\ \varphi_1\chi_2 \\ \varphi_2\chi_1 \end{pmatrix}$$

is an isomorphism of representations, i.e. the diagram

$$\begin{tikzcd} \mathbb{C}(4) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\mathbb{C}^4) \arrow[d, "\wr"] \\ \mathbb{C}(2) \hat{\otimes} \mathbb{C}(2) \arrow[r, "\sim"] & \mathrm{End}(\mathbb{C}^2 \hat{\otimes} \mathbb{C}^2) \end{tikzcd}$$

commutes, where the right-hand arrow is induced by our map $\mathbb{C}^2 \hat{\otimes} \mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^4$.

Proof. Calculation. □

Since $\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_2$ is naturally identified with $\mathbb{C}l_4$, choices of isomorphisms $\mathbb{C}l_2 \xrightarrow{\sim} \mathbb{C}(2)$ and $\mathbb{C}(2) \hat{\otimes} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$ must induce an isomorphism $\mathbb{C}l_4 \xrightarrow{\sim} \mathbb{C}(4)$.

Proposition 4.1.4. *There is an isomorphism*

$$\Delta_2 \hat{\otimes} \Delta_2 \xrightarrow{\sim} \Delta_4$$

such that

$$\begin{tikzcd} \mathbb{C}l_4 \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta_4) \arrow[d, "\wr"] \\ \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_2 \arrow[r, "\sim"] & \mathrm{End}(\Delta_2 \hat{\otimes} \Delta_2) \end{tikzcd}$$

commutes.

Proof. Follows from Proposition 4.1.3. □

It is not clear how to write the isomorphism of representations $\Delta_2 \hat{\otimes} \Delta_2 \xrightarrow{\sim} \Delta_4$ without reference to a frame. However, the action of $\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_2$ can be written as

$$(u \hat{\otimes} v)\varphi \hat{\otimes} \chi = u\varphi \hat{\otimes} v^{even}\chi + u\bar{\varphi} \hat{\otimes} v^{odd}\chi$$

where $\bar{\varphi}$ is the involute of φ with respect to the splitting of Δ_2 into semispinor spaces and

$$v^{even} = \frac{1}{2}(v + \bar{v}) \ , \ v^{odd} = \frac{1}{2}(v - \bar{v}) \ .$$

Knowing the formula for the action of $u \hat{\otimes} v$ *without* choice of a frame will allow us to generalise to higher dimensions in the next section.
The semispinor spaces are defined as the eigenspaces of the complex volume element $\omega_2 = -ie_1e_2 \in \mathbb{C}l_2$, so we have

$$\bar{\varphi} = \omega_2\varphi.$$

which will be useful later.
Clifford multiplication by an element v of the *first* copy of $\mathbb{R}^2$ is given by

$$\mathbb{C}l_2 \supset \mathbb{R}^2 \ni v \to v \hat{\otimes} 1 \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_2 \ , \ (v \hat{\otimes} 1) \cdot \varphi \hat{\otimes} \chi = (v \cdot \varphi) \hat{\otimes} \chi$$

and if v is in the *second* copy of $\mathbb{R}^2$

$$\mathbb{C}l_2 \supset \mathbb{R}^2 \ni v \to 1 \mathbin{\hat{\otimes}} v \in \mathbb{C}l_2 \mathbin{\hat{\otimes}} \mathbb{C}l_2 \ , \ (1 \mathbin{\hat{\otimes}} v) \cdot \varphi \mathbin{\hat{\otimes}} \chi = \bar{\varphi} \mathbin{\hat{\otimes}} (v \cdot \chi) \ .$$

4.2 Even plus even dimensions

Suppose we are in a more general situation: we are given the complex Clifford representations associated to $\mathbb{R}^{2k}$ and $\mathbb{R}^{2l}$ (for $k, l \geq 1$) and we want to know how to combine them to form the Clifford representation associated to $\mathbb{R}^{2k} \oplus \mathbb{R}^{2l}$. Well, $\mathbb{C}l_{2k}$ has a unique irreducible complex representation Δ_{2k} of complex dimension 2^k which possesses an involution whose action is just that of the complex volume element in $\mathbb{C}l_{2k}$ and is represented by the block matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the splitting of the spin representation

$$\Delta^0_{2k} \stackrel{\text{def}}{=} \Delta_{2k}|_{\mathbb{C}l^0_{2k}} = \Delta^{0+}_{2k} \oplus \Delta^{0-}_{2k}$$

into semispinors. Similarly for $\mathbb{C}l_{2l}$ and Δ_{2l}. There is a natural isomorphism

$$\mathbb{C}l_{2k} \mathbin{\hat{\otimes}} \mathbb{C}l_{2l} \stackrel{\text{canon}}{\cong} \mathbb{C}l_{2k+2l} \ .$$

Definition 4.2.1. *The $\mathbb{Z}_2$-graded algebra $\mathbb{C}l_{2k} \mathbin{\hat{\otimes}} \mathbb{C}l_{2l}$ acts on the space $\Delta_{2k} \otimes \Delta_{2l}$ by*

$$(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \chi \stackrel{\text{def}}{=} u\varphi \mathbin{\hat{\otimes}} v^{even}\chi + u\bar{\varphi} \mathbin{\hat{\otimes}} v^{odd}\chi$$

which defines a $\mathbb{Z}_2$-graded irreducible complex representation $\Delta_{2k} \mathbin{\hat{\otimes}} \Delta_{2l}$. Notice that this action can also be written as

$$(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \chi = (-1)^{|v||\varphi|} u\varphi \mathbin{\hat{\otimes}} v\chi$$

when v and φ are of pure grades $|v|$ and $|\varphi|$ respectively. Clifford multiplication is given by the same formulae as in Section 4.1.

Remark 4.2.2. *The use of the symbol $\hat{\otimes}$ is due to the fact that this is a $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded representations, and here this can be seen simply by the alternative expression for the action. We will abuse this notation in the other cases—see Remark 4.3.2.*

Proposition 4.2.3. *There exists an isomorphism*

$$\Delta_{2k} \mathbin{\hat{\otimes}} \Delta_{2l} \xrightarrow{\sim} \Delta_{2k+2l}$$

such that the diagram

$$\begin{tikzcd}
\mathbb{C}l_{2k+2l} \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta_{2k+2l}) \arrow[d, "\wr"] \\
\mathbb{C}l_{2k} \mathbin{\hat{\otimes}} \mathbb{C}l_{2l} \arrow[r, "\sim"] & \mathrm{End}(\Delta_{2k} \mathbin{\hat{\otimes}} \Delta_{2l})
\end{tikzcd}$$

commutes, where the right-hand arrow is induced by the isomorphism of $\Delta_{2k} \hat{\otimes} \Delta_{2l}$ with Δ_{2k+2l}.

Proof. This follows from uniqueness of the irreducible Clifford representations. □

It is not clear how to write the isomorphism explicitly without using a frame for the (high-dimensional) Clifford representations.

Denoting the restriction of Δ_{2k+2l} to the even part

$$\mathbb{C}l^0_{2k+2l} = \left(\mathbb{C}l^0_{2k} \hat{\otimes} \mathbb{C}l^0_{2l}\right) + \left(\mathbb{C}l^1_{2k} \hat{\otimes} \mathbb{C}l^1_{2l}\right)$$

by Δ^0_{2k+2l}, we have a splitting into irreducibles

$$\begin{aligned}
\Delta^{0+}_{2k+2l} &= (\Delta_{2k} \hat{\otimes} \Delta_{2l})^{0+} = (\Delta^{0+}_{2k} \hat{\otimes} \Delta^{0+}_{2l}) + (\Delta^{0-}_{2k} \hat{\otimes} \Delta^{0-}_{2l})\ , \\
\Delta^{0-}_{2k+2l} &= (\Delta_{2k} \hat{\otimes} \Delta_{2l})^{0-} = (\Delta^{0+}_{2k} \hat{\otimes} \Delta^{0-}_{2l}) + (\Delta^{0-}_{2k} \hat{\otimes} \Delta^{0+}_{2l})
\end{aligned}$$

where the $\pm$ summands are given by the ± 1-eigenspaces of the action of the respective complex volume elements.

4.3 Two plus three dimensions

As we did for the situation with two copies of $\mathbb{R}^2$, we shall describe the complex Clifford representation of the sum of a copy of $\mathbb{R}^2$ and a copy of $\mathbb{R}^3$ in terms of their associated Clifford representations. We have non-canonical isomorphisms

$$\mathbb{C}l_2 \cong \mathbb{C}(2)\ ,\ \mathbb{C}l_3 \cong \mathbb{C}(2) \oplus \mathbb{C}(2)$$

which can be chosen in such a way that the involutions are given by

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

and

$$\overline{\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]} = \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] .$$

Moreover these involutions are intertwined by the natural inclusion of $\mathbb{C}l_2$ inside $\mathbb{C}l_3$ induced by $\mathbb{R}^2 \subset \mathbb{R}^3$ (as the *first* two dimensions). The complex algebra $\mathbb{C}l_3$ has two irreducible complex representations, denoted Δ_3^+ and Δ_3^-, defined by the projections $\mathbb{C}l_3 \to \mathbb{C}l_3^\pm$ where $\mathbb{C}l_3^\pm$ is the ± 1-eigenspace of the action of the complex volume element $\omega_3 = -e_1e_2e_3 \in \mathbb{C}l_3$.

There is a natural isomorphism

$$\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \overset{\text{canon}}{\cong} \mathbb{C}l_5$$

and $\mathbb{C}l_5$ is non-canonically isomorphic to the complex algebra $\mathbb{C}(4)\oplus\mathbb{C}(4)$. It has two irreducible complex representations Δ_5^+ and Δ_5^- defined exactly as $\Delta_3^\pm$ but using $\omega_5 = -ie_1e_2e_3e_4e_5 \in \mathbb{C}l_5$.

The multiplication on $\mathbb{C}(2) \mathbin{\hat{\otimes}} \big[\mathbb{C}(2) \oplus \mathbb{C}(2)\big]$ is written as

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] \right\} \cdot \left\{ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] \right\}$$

$$\begin{aligned} = \Bigg\{ &\frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix}, \begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \right] \\ &+ \frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A-\alpha & B-\beta \\ C-\gamma & D-\delta \end{pmatrix}, \begin{pmatrix} \alpha-A & \beta-B \\ \gamma-C & \delta-D \end{pmatrix} \right] \Bigg\} \end{aligned}$$

$$\cdot \left\{ \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] \right\}$$

$$\begin{aligned} = &\frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] \\ + &\frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] \\ + &\frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A-\alpha & B-\beta \\ C-\gamma & D-\delta \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} \alpha-A & \beta-B \\ \gamma-C & \delta-D \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right] \\ - &\frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A-\alpha & B-\beta \\ C-\gamma & D-\delta \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \begin{pmatrix} \alpha-A & \beta-B \\ \gamma-C & \delta-D \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \right]. \end{aligned}$$

The algebra $\mathbb{C}(2) \mathbin{\hat{\otimes}} \big[\mathbb{C}(2) \oplus \mathbb{C}(2)\big]$ is isomorphic to $\mathbb{C}(4) \oplus \mathbb{C}(4)$, but we do not include an explicit formula as we did for $\mathbb{C}(2) \mathbin{\hat{\otimes}} \mathbb{C}(2) \xrightarrow{\sim} \mathbb{C}(4)$ as it will be rather cumbersome.

Since $\mathbb{C}(2) \mathbin{\hat{\otimes}} \big[\mathbb{C}(2) \oplus \mathbb{C}(2)\big]$ is isomorphic to $\mathbb{C}(4) \oplus \mathbb{C}(4)$, it must have two irreducible complex representations.

Definition 4.3.1. *The complex representation* $\mathbb{C}^2 \mathbin{\hat{\otimes}} \big[\mathbb{C}^2 \oplus \mathbb{C}^2\big]$ *of* $\mathbb{C}(2) \mathbin{\hat{\otimes}} \big[\mathbb{C}(2) \oplus \mathbb{C}(2)\big]$ *acting on* $\mathbb{C}^2 \otimes \big[\mathbb{C}^2 \oplus \mathbb{C}^2\big]$ *is defined by*

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \otimes \left[\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right] \\ &\overset{\text{def}}{=} \frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \otimes \left[\begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \begin{pmatrix} A+\alpha & B+\beta \\ C+\gamma & D+\delta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right] \\ &+ \frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi_1 \\ -\varphi_2 \end{pmatrix} \otimes \left[\begin{pmatrix} A-\alpha & B-\beta \\ C-\gamma & D-\delta \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \begin{pmatrix} \alpha-A & \beta-B \\ \gamma-C & \delta-D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right]. \end{aligned}$$

We also use $\hat{\otimes}$ *for its elements.*

A calculation shows this is indeed a representation. We obtained the above formula for the action by breaking the element of $\big[\mathbb{C}^2 \oplus \mathbb{C}^2\big]$ into even and odd parts, and breaking (φ_1, φ_2) into even and odd parts, then acting as if we were just multiplying two elements of a $\mathbb{Z}_2$-graded product of algebras. The result simplifies to the above.

Remark 4.3.2. *Although we use the symbol* $\hat{\otimes}$*, this is not a* $\mathbb{Z}_2$*-graded tensor product of* $\mathbb{Z}_2$*-graded representations because* $\mathbb{C}^2 \oplus \mathbb{C}^2$ *is not a* $\mathbb{Z}_2$*-graded representation of* $\mathbb{C}l_3$*. However, in this setting, this is about as close as we can get to a* $\mathbb{Z}_2$*-graded product and so we shall continue to use* $\hat{\otimes}$*.*

Proposition 4.3.3. *There exists an isomorphism* $\mathbb{C}^4 \oplus \mathbb{C}^4 \xrightarrow{\sim} \mathbb{C}^2 \mathbin{\hat{\otimes}} \left[\mathbb{C}^2 \oplus \mathbb{C}^2\right]$ *such that*

$$\begin{tikzcd}
\mathbb{C}(4) \oplus \mathbb{C}(4) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\mathbb{C}^4) \oplus \mathrm{End}(\mathbb{C}^4) \arrow[d] \\
\mathbb{C}(2) \mathbin{\hat{\otimes}} \left[\mathbb{C}(2) \oplus \mathbb{C}(2)\right] \arrow[r] & \mathrm{End}(\mathbb{C}^2 \mathbin{\hat{\otimes}} \left[\mathbb{C}^2 \oplus \mathbb{C}^2\right])
\end{tikzcd}$$

commutes.

Proof. This follows from the definition. ☐

In the notation of Definition 4.3.1, the conditions

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \ , \ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

each define a subspace of $\mathbb{C}^2 \mathbin{\hat{\otimes}} \left[\mathbb{C}^2 \oplus \mathbb{C}^2\right]$ invariant under the action; the irreducible summands are

$$\mathbb{C}^2 \mathbin{\hat{\otimes}} \left[\mathbb{C}^2 \oplus 0\right] \text{ and } \mathbb{C}^2 \mathbin{\hat{\otimes}} \left[0 \oplus \mathbb{C}^2\right] \ .$$

Now that we have understood this with matrices, we can restate it using more abstract notation. Since $\mathbb{C}l_2 \mathbin{\hat{\otimes}} \mathbb{C}l_3$ is naturally identified with $\mathbb{C}l_5$, choices of isomorphisms $\mathbb{C}l_2 \xrightarrow{\sim} \mathbb{C}(2)$, $\mathbb{C}l_3 \xrightarrow{\sim} \mathbb{C}(2) \oplus \mathbb{C}(2)$, $\mathbb{C}(2) \mathbin{\hat{\otimes}} \left[\mathbb{C}(2) \oplus \mathbb{C}(2)\right] \xrightarrow{\sim} \mathbb{C}(4) \oplus \mathbb{C}(4)$ must induce an isomorphism $\mathbb{C}l_5 \xrightarrow{\sim} \mathbb{C}(4) \oplus \mathbb{C}(4)$.

Proposition 4.3.4. *There exists an isomorphism* $\Delta_5^+ \oplus \Delta_5^- \xrightarrow{\sim} \Delta_2 \mathbin{\hat{\otimes}} (\Delta_3^+ \oplus \Delta_3^-)$ *such that*

$$\begin{tikzcd}
\mathbb{C}l_5 \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta_5^+) \oplus \mathrm{End}(\Delta_5^-) \arrow[d, "\wr"] \\
\mathbb{C}l_2 \mathbin{\hat{\otimes}} \mathbb{C}l_3 \arrow[r, "\sim"] & \mathrm{End}(\Delta_2 \mathbin{\hat{\otimes}} \Delta_3^+) \oplus \mathrm{End}(\Delta_2 \mathbin{\hat{\otimes}} \Delta_3^-)
\end{tikzcd}$$

commutes, where $\Delta_2 \mathbin{\hat{\otimes}} (\Delta_3^+ \oplus \Delta_3^-)$ *has been split into its irreducible summands* $\Delta_2 \mathbin{\hat{\otimes}} \Delta_3^+$ *and* $\Delta_2 \mathbin{\hat{\otimes}} \Delta_3^-$.

Proof. This is obvious from Proposition 4.3.3. ☐

The action of $\mathbb{C}l_2 \mathbin{\hat{\otimes}} \mathbb{C}l_3$ on $\Delta_2 \mathbin{\hat{\otimes}} (\Delta_3^+ \oplus \Delta_3^-)$ may be written as

$$(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \left[\chi, \psi\right] = u\varphi \mathbin{\hat{\otimes}} v^{even}\left[\chi, \psi\right] + u\bar{\varphi} \mathbin{\hat{\otimes}} v^{odd}\left[\chi, \psi\right] \ .$$

Notice how similar this is to the case in Section 4.2. Equivalently

$$(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \left[\chi, \psi\right] = u\varphi^{even} \mathbin{\hat{\otimes}} v\left[\chi, \psi\right] + u\varphi^{odd} \mathbin{\hat{\otimes}} \bar{v}\left[\chi, \psi\right]$$

and equivalently again

$$(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \left[\chi, \psi\right] = (-1)^{|v||\varphi|} u\varphi \mathbin{\hat{\otimes}} v\left[\chi, \psi\right]$$

when v and φ have pure grade.
We can check how our two invariant subspaces $\Delta_2 \hat{\otimes} \Delta_3^+$ and $\Delta_2 \hat{\otimes} \Delta_3^-$ correspond to Δ_5^+ and Δ_5^-. To do this, recall that $\omega_3 = -e_1 e_2 e_3$ acts on $\Delta_3^+ \oplus \Delta_3^-$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then $\mathbb{C}l_5 \ni \omega_5 = -ie_1e_2e_3e_4e_5$ maps to

$$\begin{aligned} -i(e_1 \hat{\otimes} 1)(e_2 \hat{\otimes} 1)(1 \hat{\otimes} e_1)(1 \hat{\otimes} e_2)(1 \hat{\otimes} e_3) \\ = -ie_1e_2 \hat{\otimes} e_1e_2e_3 \\ = -\omega_2 \hat{\otimes} \omega_3 \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \ , \end{aligned}$$

which acts on $\Delta_2 \hat{\otimes} \Delta_3^+$ as

$$(-\omega_2 \hat{\otimes} \omega_3)\varphi \hat{\otimes} \big[\chi, 0\big] = -\omega_2\bar{\varphi} \hat{\otimes} \omega_3\big[\chi, 0\big] = -\varphi \hat{\otimes} \big[\chi, 0\big]$$

and on $\Delta_2 \hat{\otimes} \Delta_3^-$ as

$$(-\omega_2 \hat{\otimes} \omega_3)\varphi \hat{\otimes} \big[0, \psi\big] = -\omega_2\bar{\varphi} \hat{\otimes} \omega_3\big[0, \psi\big] = \varphi \hat{\otimes} \big[0, \psi\big]$$

where we have used $\omega_2\varphi = \bar{\varphi}$. Thus the summands correspond as

$$\Delta_2 \hat{\otimes} \Delta_3^+ \cong \Delta_5^- \ , \ \Delta_2 \hat{\otimes} \Delta_3^- \cong \Delta_5^+ \ .$$

4.4 Even plus odd dimensions

Now consider the case of arbitrary even and odd dimensions respectively: we are given the complex Clifford representations associated to $\mathbb{R}^{2k}$ and $\mathbb{R}^{2l+1}$ (for $k, l \geq 1$) and we want to know how to write the complex Clifford representation of the sum in terms of these. The same comments about $\mathbb{C}l_{2k}$ and Δ_{2k} made in Section 4.2 remain valid, and $\mathbb{C}l_{2l+1}$ has two irreducible complex representations Δ_{2l+1}^+ and Δ_{2l+1}^- each of dimension 2^l. They do not possess involutions, but here we make a useful observation: the action of $\mathbb{C}l_{2k} \hat{\otimes} \mathbb{C}l_{2l}$ on $\Delta_{2k} \hat{\otimes} \Delta_{2l}$ does not require the use of the involution on Δ_{2l} but only of that on Δ_{2k}.
There is a natural isomorphism

$$\mathbb{C}l_{2k} \hat{\otimes} \mathbb{C}l_{2l+1} \overset{\text{canon}}{\cong} \mathbb{C}l_{2k+2l+1} \ .$$

Definition 4.4.1. *The $\mathbb{Z}_2$-graded algebra $\mathbb{C}l_{2k} \hat{\otimes} \mathbb{C}l_{2l+1}$ acts on the space $\Delta_{2k} \otimes (\Delta_{2l+1}^+ \oplus \Delta_{2l+1}^-)$ by*

$$(u \hat{\otimes} v)\varphi \hat{\otimes} \big[\chi, \psi\big] = u\varphi \hat{\otimes} v^{even}\big[\chi, \psi\big] + u\bar{\varphi} \hat{\otimes} v^{odd}\big[\chi, \psi\big]$$

which defines a reducible complex representation $\Delta_{2k} \hat{\otimes} (\Delta_{2l+1}^+ \oplus \Delta_{2l+1}^-)$.

Remark 4.4.2. *Although we use the symbol $\hat{\otimes}$, this is not a $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded representations because $\Delta_{2l+1}^+ \oplus \Delta_{2l+1}^-$ is not a $\mathbb{Z}_2$-graded representation of $\mathbb{C}l_{2l+1}$. However, in this setting, this is about as close as we can get to a $\mathbb{Z}_2$-graded product and so we shall continue to use $\hat{\otimes}$.*

Proposition 4.4.3. *There exists an isomorphism* $\Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1} \oplus \Delta^-_{2l+1}) \xrightarrow{\sim} \Delta^+_{2k+2l+1} \oplus \Delta^-_{2k+2l+1}$ *such that*

$$\begin{tikzcd}
\mathbb{C}l_{2k+2l+1} \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta^+_{2k+2l+1}) \oplus \mathrm{End}(\Delta^-_{2k+2l+1}) \arrow[d, "\wr"] \\
\mathbb{C}l_{2k} \hat{\otimes} \mathbb{C}l_{2l+1} \arrow[r, "\sim"] & \mathrm{End}(\Delta_{2k} \hat{\otimes} \Delta^+_{2l+1}) \oplus \mathrm{End}(\Delta_{2k} \hat{\otimes} \Delta^+_{2l+1})
\end{tikzcd}$$

commutes, which restricts to isomorphisms of the irreducible summands

$$\Delta_{2k} \hat{\otimes} \Delta^+_{2l+1} \xrightarrow{\sim} \Delta^-_{2k+2l+1}$$

and

$$\Delta_{2k} \hat{\otimes} \Delta^-_{2l+1} \xrightarrow{\sim} \Delta^+_{2k+2l+1} \ .$$

Proof. This is obvious. $\square$

Again, an explicit formula for the isomorphism between Clifford representations is not clear. Clifford multiplication is given by the same formulae as in Section 4.2, with χ replaced by $\big[\chi, \psi\big]$. That is, Clifford multiplication by an element v in $\mathbb{R}^{2k} \subset \mathbb{R}^{2k} \oplus \mathbb{R}^{2l+1}$ is given by

$$\mathbb{C}l_k \supset \mathbb{R}^{2k} \ni v \to v \hat{\otimes} 1 \in \mathbb{C}l_k \hat{\otimes} \mathbb{C}l_{2l+1} \ , \ (v \hat{\otimes} 1) \cdot \varphi \hat{\otimes} \big[\chi, \psi\big] = (v \cdot \varphi) \hat{\otimes} \big[\chi, \psi\big]$$

and if v is in $\mathbb{R}^{2l+1} \subset \mathbb{R}^{2k} \oplus \mathbb{R}^{2l+1}$

$$\mathbb{C}l_{2l+1} \supset \mathbb{R}^{2l+1} \ni v \to 1 \hat{\otimes} v \in \mathbb{C}l_k \hat{\otimes} \mathbb{C}l_{2l+1} \ , \ (1 \hat{\otimes} v) \cdot \varphi \hat{\otimes} \big[\chi, \psi\big] = \bar{\varphi} \hat{\otimes} \big[v \cdot \chi, v \cdot \psi\big] \ .$$

When we restrict $\Delta^+_{2k+2l+1}$ and $\Delta^-_{2k+2l+1}$ to the even part

$$\mathbb{C}l^0_{2k+2l+1} = \big(\mathbb{C}l^0_{2k} \hat{\otimes} \mathbb{C}l^0_{2l+1}\big) + \big(\mathbb{C}l^1_{2k} \hat{\otimes} \mathbb{C}l^1_{2l+1}\big)$$

they remain irreducible and become isomorphic.

4.5 Three plus three dimensions

This is the hardest case. If $\mathcal{V}^{even}$ is a complex bilinear form space, we can decompose $\mathcal{V}$ into a sum of maximally isotropic subspaces

$$\mathcal{V} = \mathcal{U} + \mathcal{U}^*$$

where $\mathcal{U}^*$ is identified with the dual of $\mathcal{U}$ using the bilinear form. It is a well-known construction that the unique irreducible complex representation of $\mathbb{C}l(\mathcal{V})$ is $\Lambda\mathcal{U}$, with Clifford multiplication given by

$$u \cdot \chi = u \wedge \chi \ , \quad u^* \cdot \chi = -p u \lrcorner \chi$$

for $u \in \mathcal{U}$, $u^* = \langle \cdot, u \rangle \in \mathcal{U}^*$ and $\chi \in \Lambda^p\mathcal{U}$. With this in mind, we now describe a similar construction and use it to extend the bilinear form space by a dimension.

Lemma 4.5.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be complex bilinear form spaces and let $e_0 \in \mathcal{W}$ have unit norm. Let $\mathcal{W}^{even} = \mathbb{C}e_0 \oplus \mathcal{V}^{odd}$ and $\mathcal{U} \subset \mathcal{V}$ be maximally isotropic so that we can write[2] $\mathcal{V} = \mathbb{C}e_1 \oplus (\mathcal{U} + \mathcal{U}^*)$. We further demand that appending e_0 to an oriented frame of $\mathcal{V}$ must give an oriented frame of $\mathcal{W}$. Then the complex irreducible representation of $\mathbb{C}l(\mathcal{W})$ is $\Lambda\mathcal{U} \oplus \Lambda\mathcal{U}$ (where the splitting is preserved by $\mathbb{C}l^0(\mathcal{W})$) and with respect to this splitting, the formulae*

$$v \cdot_{\mathcal{W}} = \begin{pmatrix} 0 & -\mu v \cdot_{\mathcal{V}} \\ -(1/\mu) v \cdot_{\mathcal{V}} & 0 \end{pmatrix}$$

when v is odd and $\cdot_{\mathcal{V}}$, $\cdot_{\mathcal{W}}$ are the respective Clifford multiplications corresponding to $\mathcal{V}$, $\mathcal{W}$, and

$$e_0 \cdot_{\mathcal{W}} = -i^{\dim \mathcal{W}} \begin{pmatrix} 0 & \mu \\ -1/\mu & 0 \end{pmatrix}$$

determine the action of $\mathbb{C}l(\mathcal{W})$ entirely, for each choice of $\mu \in \mathbb{C} - \{0\}$. Different choices of μ yield isomorphic representations. We can put $\mu = -1$, whence

$$v \cdot_{\mathcal{W}} = \begin{pmatrix} 0 & v \cdot_{\mathcal{V}} \\ v \cdot_{\mathcal{V}} & 0 \end{pmatrix}$$

when v is odd and

$$e_0 \cdot_{\mathcal{W}} = -i^{\dim \mathcal{W}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Proof. Calculation. ∎

Remark 4.5.2. *Note that since* $\dim \mathcal{W}$ *is even, the factor* $-i^{\dim \mathcal{W}}$ *is quite simple:*

$$-i^{\dim \mathcal{W}} = \begin{cases} +1 & \dim \mathcal{W} = 2 \bmod 4 \\ -1 & \dim \mathcal{W} = 0 \bmod 4 \end{cases} .$$

Lemma 4.5.1 tells us that the space on which we want $\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \subset \mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3$ to act should be $\Delta_2 \hat{\otimes} (\Delta_3^+ \oplus \Delta_3^+)$, where *odd* elements of $\mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \subset \mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3$ act in the same way as they do in Section 4.4, but now swapping the summands around. We have used Δ_3^+ twice, but we could equally well have used Δ_3^- twice instead. For $u \in \mathbb{C}l_2$ and $v \in \mathbb{C}l_3$ odd, we must have

$$\begin{aligned} (u \hat{\otimes} 1)\varphi \hat{\otimes} \left[\chi, \psi\right] &= u\varphi \hat{\otimes} \left[\psi, \chi\right] , \\ (1 \hat{\otimes} v)\varphi \hat{\otimes} \left[\chi, \psi\right] &= \bar{\varphi} \hat{\otimes} \left[v\psi, v\chi\right] . \end{aligned}$$

From these we see that for $u \in \mathbb{C}l_2$ and $v \in \mathbb{C}l_3$ even, we get

$$\begin{aligned} (u \hat{\otimes} 1)\varphi \hat{\otimes} \left[\chi, \psi\right] &= u\varphi \hat{\otimes} \left[\chi, \psi\right] , \\ (1 \hat{\otimes} v)\varphi \hat{\otimes} \left[\chi, \psi\right] &= \varphi \hat{\otimes} \left[v\chi, v\psi\right] . \end{aligned}$$

A general simple element $u \hat{\otimes} v \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3$ can be written as

$$u \hat{\otimes} v = u^{even} \hat{\otimes} v^{even} + u^{even} \hat{\otimes} v^{odd} + u^{odd} \hat{\otimes} v^{even} + u^{odd} \hat{\otimes} v^{odd} ,$$

whence we obtain the general formula for the action.

[2] We reserve the $\oplus$ symbol for when the sum is orthogonal, and use $+$ otherwise.

Definition 4.5.3. *The $\mathbb{Z}_2$-graded algebra $\mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3$ acts on the space $\Delta_2 \otimes (\Delta_3^- \oplus \Delta_3^-)$ in the following way. An element $u \hat{\otimes} v \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \subset \mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3$ acts as*

$$\begin{aligned}(u \hat{\otimes} v)\varphi \hat{\otimes} \big[\chi, \psi\big] \overset{\text{def}}{=} & u^{even}\varphi \hat{\otimes} \big[v^{even}\chi, v^{even}\psi\big] + u^{even}\bar{\varphi} \hat{\otimes} \big[v^{odd}\psi, v^{odd}\chi\big] \\ & + u^{odd}\varphi \hat{\otimes} \big[v^{even}\psi, v^{even}\chi\big] + u^{odd}\bar{\varphi} \hat{\otimes} \big[v^{odd}\chi, v^{odd}\psi\big] \ .\end{aligned}$$

and the extra vector $e_3 \hat{\otimes} 1$ acts as

$$(e_3 \hat{\otimes} 1)\varphi \hat{\otimes} \big[\chi, \psi\big] = \varphi \hat{\otimes} \big[-\psi, \chi\big] \ .$$

This defines a $\mathbb{Z}_2$-graded irreducible complex representation[3] $\Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+)$.

We can check that this well-defined; indeed, a calculation shows that for $u_1, u_2 \in \mathbb{C}l_2$ and $v_1, v_2 \in \mathbb{C}l_3$

$$\Big[(u_1 \hat{\otimes} v_1)(u_2 \hat{\otimes} v_2)\Big]\varphi \hat{\otimes} \big[\chi, \psi\big] = (u_1 \hat{\otimes} v_1)\Big[(u_2 \hat{\otimes} v_2)\varphi \hat{\otimes} \big[\chi, \psi\big]\Big]$$

and also, since $(u \hat{\otimes} v)(e_3 \hat{\otimes} 1) = (e_3 \hat{\otimes} 1)(\bar{u} \hat{\otimes} \bar{v})$,

$$(u \hat{\otimes} v)\Big[(e_3 \hat{\otimes} 1)\varphi \hat{\otimes} \big[\chi, \psi\big]\Big] = (e_3 \hat{\otimes} 1)\Big[(\bar{u} \hat{\otimes} \bar{v})\varphi \hat{\otimes} \big[\chi, \psi\big]\Big] \ .$$

These conditions verify that our formulae do define a representation of $\mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3$ on $\Delta_2 \hat{\otimes} (\Delta_3^+ \oplus \Delta_3^+)$. Moreover, this space has eight complex dimensions, so must be the unique irreducible complex representation of $\mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3 = \mathbb{C}l_6$. It would be nice to write this action without reference to the particular element e_3, but it is not clear how to do so.

Remark 4.5.4. *Although we use the symbol $\hat{\otimes}$, this is certainly not a $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded representations because* neither *factor is a $\mathbb{Z}_2$-graded representation of its respective Clifford algebra. However, in this setting, this is about as close as we can get to a $\mathbb{Z}_2$-graded product and so we shall continue to use $\hat{\otimes}$.*

Proposition 4.5.5. *There exists an isomorphism $\Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+) \xrightarrow{\sim} \Delta_6$ such that*

$$\begin{tikzcd} \mathbb{C}l_6 \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta_6) \arrow[d, "\wr"] \\ \mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3 \arrow[r, "\sim"] & \mathrm{End}\big(\Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+)\big) \end{tikzcd}$$

commutes. Similarly for $\Delta_2 \hat{\otimes} (\Delta_3^- + \Delta_3^-) \cong \Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+)$.

Proof. This follows from uniqueness of the irreducible Clifford representations. $\square$

Clifford multiplication by an element v of the copy of $\mathbb{R}^2$ is given by

$$\mathbb{C}l_2 \supset \mathbb{R}^2 \ni v \to v \hat{\otimes} 1 \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \ , \ (v \hat{\otimes} 1) \cdot \varphi \hat{\otimes} \big[\chi, \psi\big] = (v \cdot \varphi) \hat{\otimes} \big[\psi, \chi\big]$$

[3]Since the summands get swapped by this action, we use the symbol $+$ instead of $\oplus$.

and if v is in the copy of $\mathbb{R}^3$,

$$\mathbb{C}l_3 \supset \mathbb{R}^3 \ni v \to 1 \hat{\otimes} v \in \mathbb{C}l_2 \hat{\otimes} \mathbb{C}l_3 \ , \ (1 \hat{\otimes} v) \cdot \varphi \hat{\otimes} \big[\chi, \psi\big] = \bar{\varphi} \hat{\otimes} \big[v \cdot \psi, v \cdot \chi\big] \ .$$

The remaining unit vector is $e_3 \hat{\otimes} 1$, whose action can be found in Definition 4.5.3.
Do not be fooled into thinking that the two copies of Δ_3^+ in $\Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+)$ behave in the same way. The complex volume form in six dimensions can be written as

$$\begin{aligned}\mathbb{C}l_6 \ni \omega_6 = ie_1e_2e_3e_4e_5e_6 \to i(e_1 \hat{\otimes} 1)(e_2 \hat{\otimes} 1)(e_3 \hat{\otimes} 1)(1 \hat{\otimes} e_1)(1 \hat{\otimes} e_2)(1 \hat{\otimes} e_3)\\ = (e_3 \hat{\otimes} 1)(-ie_1e_2) \hat{\otimes} (-e_1e_2e_3)\\ = (e_3 \hat{\otimes} 1)(\omega_2 \hat{\otimes} \omega_3) \in \mathbb{C}l_3 \hat{\otimes} \mathbb{C}l_3 \ ,\end{aligned}$$

which acts on $\varphi \hat{\otimes} \big[\chi, \psi\big] \in \Delta_2 \hat{\otimes} (\Delta_3^+ + \Delta_3^+)$ as

$$\begin{aligned}&(e_3 \hat{\otimes} 1)(\omega_2 \hat{\otimes} \omega_3)\varphi \hat{\otimes} \big[\chi, \psi\big]\\ &\quad = (e_3 \hat{\otimes} 1)\omega_2\bar{\varphi} \hat{\otimes} \big[\omega_3\psi, \omega_3\chi\big]\\ &\quad = (e_3 \hat{\otimes} 1)\varphi \hat{\otimes} \big[\psi, \chi\big]\\ &\quad = \varphi \hat{\otimes} \big[-\chi, \psi\big] \ .\end{aligned}$$

From this we see that the first copy of Δ_3^+ yields the spinors of negative chirality and the second copy gives the positive ones.

4.6 Odd plus odd dimensions

Finally we state the results for the general case when both dimensions are odd. There is a natural isomorphism

$$\mathbb{C}l_{2k+1} \hat{\otimes} \mathbb{C}l_{2l+1} \overset{\text{canon}}{\cong} \mathbb{C}l_{2k+2l+2} \ .$$

The definition is the same as in Section 4.5.

Definition 4.6.1. *The $\mathbb{Z}_2$-graded algebra $\mathbb{C}l_{2k+1} \hat{\otimes} \mathbb{C}l_{2l+1}$ acts on the space $\Delta_{2k} \otimes (\Delta_{2l+1}^+ \oplus \Delta_{2l+1}^+)$ in the following way. An element $u \hat{\otimes} v \in \mathbb{C}l_{2k} \hat{\otimes} \mathbb{C}l_{2l+1} \subset \mathbb{C}l_{2k+1} \hat{\otimes} \mathbb{C}l_{2l+1}$ acts as*

$$\begin{aligned}(u \hat{\otimes} v)\varphi \hat{\otimes} \big[\chi, \psi\big] \overset{\text{def}}{=} u^{even}\varphi \hat{\otimes} \big[v^{even}\chi, v^{even}\psi\big] + u^{even}\bar{\varphi} \hat{\otimes} \big[v^{odd}\psi, v^{odd}\chi\big]\\ + u^{odd}\varphi \hat{\otimes} \big[v^{even}\psi, v^{even}\chi\big] + u^{odd}\bar{\varphi} \hat{\otimes} \big[v^{odd}\chi, v^{odd}\psi\big] \ .\end{aligned}$$

and the extra vector $e_{2k+1} \hat{\otimes} 1$ acts as

$$(e_{2k+1} \hat{\otimes} 1)\varphi \hat{\otimes} \big[\chi, \psi\big] = -i^{2k+2l+2}\varphi \hat{\otimes} \big[-\psi, \chi\big] \ .$$

This defines a $\mathbb{Z}_2$-graded irreducible complex representation $\Delta_{2k} \hat{\otimes} (\Delta_{2l+1}^+ + \Delta_{2l+1}^+)$.

The comments made in Section 4.5 can be made here as well to explain why this must be the unique irreducible complex representation of $\mathbb{C}l_{2k+1} \hat{\otimes} \mathbb{C}l_{2l+1} = \mathbb{C}l_{2k+2l+2}$.

Remark 4.6.2. *Although we use the symbol $\hat{\otimes}$, this is certainly not a $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded representations because* neither *factor is a $\mathbb{Z}_2$-graded representation of its respective*

Clifford algebra. However, in this setting, this is about as close as we can get to a $\mathbb{Z}_2$-graded product and so we shall continue to use $\hat{\otimes}$.

Proposition 4.6.3. *There exists an isomorphism $\Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1} + \Delta^+_{2l+1}) \xrightarrow{\sim} \Delta_{2k+2l+2}$ such that*

$$\begin{tikzcd}
\mathbb{C}l_{2k+2l+2} \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{End}(\Delta_{2k+2l+2}) \arrow[d, "\wr"] \\
\mathbb{C}l_{2k+1} \hat{\otimes} \mathbb{C}l_{2l+1} \arrow[r, "\sim"] & \mathrm{End}\big(\Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1} + \Delta^+_{2l+1})\big)
\end{tikzcd}$$

commutes. Similarly $\Delta_{2k} \hat{\otimes} (\Delta^-_{2l+1} + \Delta^-_{2l+1}) \cong \Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1} + \Delta^+_{2l+1})$.

Proof. This follows from uniqueness of the irreducible Clifford representations. $\square$

Clifford multiplication is given by the same formulae as in Section 4.5 above.
Denoting the restriction of $\Delta_{2k+2l+2}$ to the even part

$$\mathbb{C}l^0_{2k+2l+2} = \big(\mathbb{C}l^0_{2k+1} \hat{\otimes} \mathbb{C}l^0_{2l+1}\big) + \big(\mathbb{C}l^1_{2k+1} \hat{\otimes} \mathbb{C}l^1_{2l+1}\big)$$

by $\Delta^0_{2k+2l+2}$, we have a splitting into irreducibles: when $2k+2l+2 = 0 \bmod 4$ we have

$$\begin{aligned}
\Delta^{0+}_{2k+2l+2} &= \Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1}, 0) \ , \\
\Delta^{0-}_{2k+2l+2} &= \Delta_{2k} \hat{\otimes} (0, \Delta^+_{2l+1})
\end{aligned}$$

and when $2k+2l+2 = 2 \bmod 4$ we have

$$\begin{aligned}
\Delta^{0+}_{2k+2l+2} &= \Delta_{2k} \hat{\otimes} (0, \Delta^+_{2l+1}) \ , \\
\Delta^{0-}_{2k+2l+2} &= \Delta_{2k} \hat{\otimes} (\Delta^+_{2l+1}, 0)
\end{aligned}$$

and this is proven in precisely the way we did above by looking at the action of ω_6, or in this case $\omega_{2k+2l+2}$. If we use the space $\Delta_{2k} \hat{\otimes} (\Delta^-_{2l+1} + \Delta^-_{2l+1})$ instead, then the correspondence of the summands to the chiral semispinors swaps.

Remark 4.6.4. *Since $2k$ is even, we can choose our extra element to be $e_1 \hat{\otimes} 1$ instead of $e_{2k+1} \hat{\otimes} 1$, and this makes no difference. We will choose to do this later.*

Chapter 5

An O'Neill Formula for Spinors

The well-known and foundational paper on Riemannian submersions by O'Neill [O'N66] contains the simple formula relating the covariant derivative of a vector field $\check{Y}$ on the base of a Riemannian submersion π to the covariant derivative of its lift Y to the total space:

$$\nabla_X Y = \pi^* \check{\nabla}_{\check{X}} \check{Y} + A_X Y \ .$$

We shall now derive analogous formulae for spinor fields. This will be much more difficult and will require us to break the problem into several cases, just as we did for the spin representations in the previous chapter. We start by 'bundle-ifying' the statements we made about the spin spaces so that we have the correct setting to do calculus.

5.1 Spinor bundles

This section will explain how a spinor field on the total space $\mathcal{M}$ of a submersion is constructed from a spinor field on the base space $\mathcal{B}$ and a spinor field on the fibres.

From now on we'll restrict our attention to the case when the total space $\mathcal{M}$, base space $\mathcal{B}$ and all fibres of π are spin manifolds. Moreover we'll assume that spin structures on $\mathcal{B}$ and on all the fibres have been specified, and in such a way that those on the fibres give us a $Spin_p$-bundle $Spin(\mathscr{V})$ such that

$$\begin{tikzcd}
 & Spin(\mathscr{V}) \arrow[dl] \arrow[dd] & Spin_p \arrow[l] \arrow[dd] \\
\mathcal{M} & & \\
 & SO(\mathscr{V}) \arrow[ul] & SO_p \arrow[l]
\end{tikzcd}$$

commutes. Noting the standard inclusion of $Spin_p$ in the complex Clifford algebra $\mathbb{C}l_p$,

Definition 5.1.1. *Let $\pi : \mathcal{M}^{p+n} \to \mathcal{B}^n$ be a conformal submersion between Riemannian spin manifolds. The **vertical spinor bundle** $\mathbb{S}\mathscr{V}$ is the bundle over $\mathcal{M}$ associated to $Spin(\mathscr{V})$ by the complex Clifford representation Δ_p:*

$$\mathbb{S}\mathscr{V} \overset{\text{def}}{=} Spin(\mathscr{V}) \times_{Spin_p} \Delta_p \ .$$

In the case when $\mathbb{C}l_p$ has two irreducible complex representations Δ_p^+ and Δ_p^-, their restrictions to the even part $\mathbb{C}l_p^0$ (and hence to $Spin_p$) are the same. When $\mathbb{C}l_p$ has only one irreducible complex representation, its restriction to $\mathbb{C}l_p^0$ splits as $\Delta_p^{0+} \oplus \Delta_p^{0-}$ where $\pm$ is given by the action of the complex volume element. We denote the corresponding bundle summands by $\mathbb{S}\mathscr{V}^+$ and $\mathbb{S}\mathscr{V}^-$.

Since $\mathcal{B}$ has a spin structure $Spin(\mathcal{B}) \to SO(\mathcal{B})$ and π is smooth, there exists a $Spin_n$-bundle $Spin(\mathscr{H})$ and a map $\Xi^{Spin} : Spin(\mathscr{H}) \to Spin(\mathcal{B})$ such that

$$\begin{tikzcd}
\mathcal{M} \arrow[d, "\pi"] & Spin(\mathscr{H}) \arrow[l] \arrow[d, "\Xi^{Spin}"] & \\
 & & Spin_n \arrow[ul] \arrow[dl] \\
\mathcal{B} & Spin(\mathcal{B}) \arrow[l] &
\end{tikzcd}$$

commutes.

Definition 5.1.2. *The **horizontal spinor bundle** $\mathbb{S}\mathscr{H}$ is the bundle over $\mathcal{M}$ associated to $Spin(\mathscr{H})$ by the complex Clifford representation Δ_n:*

$$\mathbb{S}\mathscr{H} \stackrel{\text{def}}{=} Spin(\mathscr{H}) \times_{Spin_n} \Delta_n \ .$$

In the case when $\mathbb{C}l_n$ has two irreducible complex representations Δ_n^+ and Δ_n^-, their restrictions to the even part $\mathbb{C}l_n^0$ (and hence to $Spin_n$) are the same. When $\mathbb{C}l_n$ has only one irreducible complex representation, its restriction to $\mathbb{C}l_n^0$ splits as $\Delta_n^{0+} \oplus \Delta_n^{0-}$ where $\pm$ is given by the action of the complex volume element. We denote the corresponding bundle summands by $\mathbb{S}\mathscr{H}^+$ and $\mathbb{S}\mathscr{H}^-$.

The groups $Spin_p$ and $Spin_n$ are double covers (and universal for $p > 2$ and $n > 2$ respectively) of SO_p and SO_n, so $Spin_p \times Spin_n$ is a four-fold cover of $SO_p \times SO_n$. The group of deck transformations is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and we denote by

$$Spin_p \times_{\mathbb{Z}_2} Spin_n$$

the quotient of $Spin_p \times Spin_n$ by the action of the diagonal copy of $\mathbb{Z}_2$ inside $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Recall that the adapted orthonormal frame bundle $SO(\mathcal{M}, \mathscr{V})$ is the fibrewise product of $SO(\mathscr{V})$ and $SO(\mathscr{H})$, which we write as

$$SO(\mathcal{M}, \mathscr{V}) = SO(\mathscr{V}) + SO(\mathscr{H}) \ .$$

Then the bundle $Spin(\mathscr{V}) + Spin(\mathscr{H})$ is a four-fold fibrewise cover of $SO(\mathcal{M}, \mathscr{V})$.

Definition 5.1.3. *The **adapted spin structure** $Spin(\mathcal{M}, \mathscr{V})$ is the quotient of $Spin(\mathscr{V}) + Spin(\mathscr{H})$ by the diagonal copy $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \mathbb{Z}_2$ acting by deck transformations on fibres:*

$$Spin(\mathcal{M}, \mathscr{V}) \stackrel{\text{def}}{=} Spin(\mathscr{V}) +_{\mathbb{Z}_2} Spin(\mathscr{H}) \ .$$

Then $Spin(\mathcal{M},\mathscr{V})$ is a principal $Spin_p \times_{\mathbb{Z}_2} Spin_n$-bundle over $\mathcal{M}$ and its natural enlargement

$$Spin(\mathcal{M}) \stackrel{\text{def}}{=} Spin(\mathcal{M},\mathscr{V}) \times_{Spin_p\times_{\mathbb{Z}_2} Spin_n} Spin_{p+n}$$

is a spin structure on $\mathcal{M}$, where $Spin_p \times_{\mathbb{Z}_2} Spin_n$ is the subgroup of $Spin_{p+n}$ given by the preimage of $SO_p \times SO_n$ under the natural map $Spin_{p+n} \to SO_{p+n}$. We shall always use this spin structure on $\mathcal{M}$.

Remark 5.1.4. *Given two associated vector bundles $P\times_G V$ and $Q\times_H W$, their tensor product is given by*

$$(P\times_G V)\otimes(Q\times_H W) = (P+Q)\times_{G\times H} V\otimes W$$

where $P+Q$ is fibrewise product of principal bundles and $G\times H$ acts on $V\otimes W$ by linear extension of its action on simple elements, i.e. on $V\times W$.

Theorem 5.1.5. *Let $\pi:\mathcal{M}\to\mathcal{B}$ be a conformal submersion between Riemannian spin manifolds with vertical distribution $\mathscr{V}$ and horizontal distribution $\mathscr{H}$. Consider the complex spinor bundle $\mathbb{S}\mathcal{M}$ of $(\mathcal{M},g)$, in the case when $\dim\mathscr{V},\dim\mathscr{H}>1$. There are four cases.*

1. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both even****: There exists a natural isomorphism*

$$\mathbb{S}\mathcal{M} \xrightarrow{\sim} \mathbb{S}\mathscr{V}\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}$$

such that $\mathbb{C}l(\mathcal{M}) \stackrel{\text{canon}}{\cong} \mathbb{C}l(\mathscr{V})\mathbin{\hat\otimes}\mathbb{C}l(\mathscr{H})$ *acts on* $\mathbb{S}\mathcal{M}$ *as*

$$(u\mathbin{\hat\otimes} v)\varphi\mathbin{\hat\otimes}\chi = u\varphi\mathbin{\hat\otimes} v^{even}\chi + u\bar\varphi\mathbin{\hat\otimes} v^{odd}\chi\ .$$

The spinor bundle $\mathbb{S}\mathcal{M}$ is reducible and the above isomorphism restricts to isomorphisms

$$\mathbb{S}\mathcal{M}^+ \xrightarrow{\sim} \mathbb{S}\mathscr{V}^+\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}^+ + \mathbb{S}\mathscr{V}^-\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}^-\ ,$$
$$\mathbb{S}\mathcal{M}^- \xrightarrow{\sim} \mathbb{S}\mathscr{V}^+\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}^- + \mathbb{S}\mathscr{V}^-\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}^+\ .$$

2. $\dim\mathscr{V}$ ***even*** *and* $\dim\mathscr{H}$ ***odd****: There exists a natural isomorphism*

$$\mathbb{S}\mathcal{M} \xrightarrow{\sim} \mathbb{S}\mathscr{V}\mathbin{\hat\otimes}\mathbb{S}\mathscr{H}$$

such that $\mathbb{C}l(\mathcal{M}) \stackrel{\text{canon}}{\cong} \mathbb{C}l(\mathscr{V})\mathbin{\hat\otimes}\mathbb{C}l(\mathscr{H})$ *acts on* $\mathbb{S}\mathcal{M}$ *as*

$$(u\mathbin{\hat\otimes} v)\varphi\mathbin{\hat\otimes}\chi = u\varphi\mathbin{\hat\otimes} v^{even}\chi + u\bar\varphi\mathbin{\hat\otimes} v^{odd}\chi\ .$$

The spinor bundle $\mathbb{S}\mathcal{M}$ is irreducible. There are two ways to interpret $v^{odd}\chi$, given by the two Clifford multiplications, which differ by a minus sign.

3. $\dim\mathscr{V}$ ***odd*** *and* $\dim\mathscr{H}$ ***even****: There exists a natural isomorphism*

$$\mathbb{S}\mathcal{M} \xrightarrow{\sim} \mathbb{S}\mathscr{H}\mathbin{\hat\otimes}\mathbb{S}\mathscr{V}$$

such that $\mathbb{C}l(\mathcal{M}) \stackrel{\text{canon}}{\cong} \mathbb{C}l(\mathscr{H})\mathbin{\hat\otimes}\mathbb{C}l(\mathscr{V})$ *acts on* $\mathbb{S}\mathcal{M}$ *as*

$$(u\mathbin{\hat\otimes} v)\chi\mathbin{\hat\otimes}\varphi = u\chi\mathbin{\hat\otimes} v^{even}\varphi + u\bar\chi\mathbin{\hat\otimes} v^{odd}\varphi\ .$$

The spinor bundle $\mathbb{S}\mathcal{M}$ is irreducible (notice we've just swapped $\mathscr{V}$ and $\mathscr{H}$ for this case). There are two ways to interpret $v^{odd}\varphi$, given by the two Clifford multiplications, which differ by a minus sign.

4. $\dim \mathscr{V}$ *and* $\dim \mathscr{H}$ ***both odd**: For each choice of locally-defined vertical unit vector field ξ on $\mathcal{U} \subset \mathcal{M}$, denote the vertical complement by $\mathscr{V}^{\perp\xi}$. Then define $Spin(\mathscr{V}^{\perp\xi}) \subset Spin(\mathscr{V})$ as the preimage of $SO(\mathscr{V}^{\perp\xi}) \subset SO(\mathscr{V})$ under $Spin(\mathscr{V}) \to SO(\mathscr{V})$, and*

$$\mathbb{S}\mathscr{V}^{\perp\xi} \stackrel{\text{def}}{=} Spin(\mathscr{V}^{\perp\xi}) \times_{Spin_{p-1}} \Delta_{p-1} \ .$$

Then for the restrictions of the bundles to $\mathcal{U}$, there exists an isomorphism of associated bundles

$$\mathbb{S}\mathcal{M} \xrightarrow{\sim} \mathbb{S}\mathscr{V}^{\perp\xi} \mathbin{\hat{\otimes}} (\mathbb{S}\mathscr{H} + \mathbb{S}\mathscr{H})$$

such that $\mathbb{C}l(\mathcal{M}^{\perp\xi}) \stackrel{\text{def}}{=} \mathbb{C}l(\mathscr{V}^{\perp\xi}) \mathbin{\hat{\otimes}} \mathbb{C}l(\mathscr{H})$ acts on $\mathbb{S}\mathcal{M}$ as

$$\begin{aligned}(u \mathbin{\hat{\otimes}} v)\varphi \mathbin{\hat{\otimes}} \big[\chi, \psi\big] = {} & u^{even}\varphi \mathbin{\hat{\otimes}} \big[v^{even}\chi, v^{even}\psi\big] + u^{even}\bar{\varphi} \mathbin{\hat{\otimes}} \big[v^{odd}\psi, v^{odd}\chi\big] \\ & + u^{odd}\varphi \mathbin{\hat{\otimes}} \big[v^{even}\psi, v^{even}\chi\big] + u^{odd}\bar{\varphi} \mathbin{\hat{\otimes}} \big[v^{odd}\chi, v^{odd}\psi\big]\end{aligned}$$

and the extra unit vector field ξ acts as

$$(\xi \mathbin{\hat{\otimes}} 1)\varphi \mathbin{\hat{\otimes}} \big[\chi, \psi\big] = -i^{\dim \mathcal{M}}\varphi \mathbin{\hat{\otimes}} \big[-\psi, \chi\big] \ .$$

These formulae give the action of $\mathbb{C}l(\mathcal{M})$. It must be remembered that the terms $v^{odd}\chi$ and $v^{odd}\psi$ have two possible interpretations corresponding to the two Clifford multiplications, differing by a minus sign, and we must use the same action for both. The spinor bundle $\mathbb{S}\mathcal{M}$ is reducible and the above isomorphism restricts to isomorphisms

$$\begin{aligned}\mathbb{S}\mathcal{M}^+ &\xrightarrow{\sim} \mathbb{S}\mathscr{V}^{\perp\xi} \mathbin{\hat{\otimes}} (\mathbb{S}\mathscr{H}, 0) \ , \\ \mathbb{S}\mathcal{M}^- &\xrightarrow{\sim} \mathbb{S}\mathscr{V}^{\perp\xi} \mathbin{\hat{\otimes}} (0, \mathbb{S}\mathscr{H}) \ .\end{aligned}$$

Remark 5.1.6. *It should be noted that there is nothing at all special about the choice of locally-defined vertical unit vector field ξ. Any choice will do, and even though one choice may be forced to have zeros elsewhere, we can always patch together such isomorphisms to look at $\mathbb{S}\mathcal{M}$ everywhere. Since we shall be concerned with calculus, we need look only at a patch of the total space $\mathcal{M}$.*

Proof. To prove *1*, we first note that $\Delta_p \mathbin{\hat{\otimes}} \Delta_n$ is a representation of $\mathbb{C}l_p \mathbin{\hat{\otimes}} \mathbb{C}l_n = \mathbb{C}l_{p+n}$ and the inclusions

$$Spin_p \times Spin_n \subset \mathbb{C}l_p^0 \times \mathbb{C}l_n^0 \subset \mathbb{C}l_p \mathbin{\hat{\otimes}} \mathbb{C}l_n = \mathbb{C}l_{p+n}$$

and

$$Spin_p \times Spin_n \subset Spin_{p+n} \subset \mathbb{C}l_{p+n}$$

are the same. This means there is no ambiguity in $\Delta_p \mathbin{\hat{\otimes}} \Delta_n$ being a representation of $Spin_p \times Spin_n$. The element $(-1,-1) \in Spin_p \times Spin_n$ acts as the identity on $\Delta_p \mathbin{\hat{\otimes}} \Delta_n$, and so $\Delta_p \mathbin{\hat{\otimes}} \Delta_n$ is a well-defined representation of $Spin_p \times_{\mathbb{Z}_2} Spin_n$. Note that $\mathbb{S}\mathscr{V}$ is not associated only to $Spin(\mathscr{V})$ but also to $Spin(\mathscr{V}) + Spin(\mathscr{H})$:

$$\mathbb{S}\mathscr{V} = \big(Spin(\mathscr{V}) + Spin(\mathscr{H})\big) \times_{Spin_p \times Spin_n} \Delta_p$$

where $Spin_p \times Spin_n$ acts on Δ_p with kernel $Spin_n$ (however, $\mathbb{S}\mathscr{V}$ is *not* associated to $Spin(\mathcal{M}, \mathscr{V})$ since $Spin_p \times_{\mathbb{Z}_2} Spin_n$ does not act on Δ_p because $(-1,-1)$ does not act as the identity). We can make similar remarks about $\mathbb{S}\mathscr{H}$.

$$\begin{aligned}
\mathbb{S}\mathscr{V} \mathbin{\hat{\otimes}} \mathbb{S}\mathscr{H} &= \Big[\big(Spin(\mathscr{V}) + Spin(\mathscr{H})\big) \times_{Spin_p \times Spin_n} \Delta_p\Big] \\
&\quad \hat{\otimes} \Big[\big(Spin(\mathscr{V}) + Spin(\mathscr{H})\big) \times_{Spin_p \times Spin_n} \Delta_n\Big] \\
&= \Big[\big(Spin(\mathscr{V}) + Spin(\mathscr{H})\big) + \big(Spin(\mathscr{V}) + Spin(\mathscr{H})\big)\Big] \\
&\quad \times_{(Spin_p \times Spin_n) \times (Spin_p \times Spin_n)} \Delta_p \mathbin{\hat{\otimes}} \Delta_n \\
&= \Big[Spin(\mathscr{V}) + Spin(\mathscr{H})\Big] \times_{Spin_p \times Spin_n} \Delta_p \mathbin{\hat{\otimes}} \Delta_n \\
&= \Big[Spin(\mathscr{V}) +_{\mathbb{Z}_2} Spin(\mathscr{H})\Big] \times_{Spin_p \times_{\mathbb{Z}_2} Spin_n} \Delta_p \mathbin{\hat{\otimes}} \Delta_n \\
&= Spin(\mathcal{M}, \mathscr{V}) \times_{Spin_p \times_{\mathbb{Z}_2} Spin_n} \Delta_p \mathbin{\hat{\otimes}} \Delta_n \\
&= Spin(\mathcal{M}) \times_{Spin_{p+n}} \Delta_{p+n} \\
&= \mathbb{S}\mathcal{M}
\end{aligned}$$

where we've used Proposition 4.2.3. Similar remarks suffice to prove *2* and *3*. For *4*, we need a little more. The locally-defined vertical unit vector field ξ plays the role of our element $e_{2k+1} \mathbin{\hat{\otimes}} 1$, and any isomorphism $\Delta_{2k} \mathbin{\hat{\otimes}} (\Delta^+_{2l+1} + \Delta^+_{2l+1}) \xrightarrow{\sim} \Delta_{2k+2l+2}$ is unique up to scalars by Schur's Lemma. On the patch $\mathcal{U} \subset \mathcal{M}$, the result now follows by Proposition 4.6.3. $\square$

In *4* of Theorem 5.1.5 above we explained that we have to make a choice of locally-defined vertical unit vector field ξ to understand the complex spinor bundle of $\mathcal{M}$. The field ξ acts on $\mathbb{S}\mathcal{M}$ in a way different from its vertical colleagues. Note that as associated vector bundles to $Spin(\mathscr{V}^{\perp\xi})$, $\mathbb{S}\mathscr{V}^{\perp\xi}$ and $\mathbb{S}\mathscr{V}$ are naturally isomorphic. If

$$\iota_{\perp\xi} \;:\; Spin(\mathscr{V}^{\perp\xi}) \to Spin(\mathscr{V})$$

is natural inclusion, and

$$\alpha \;:\; \Delta_{p-1} \to \Delta_p$$

is the natural isomorphism of vector spaces when p is odd then

$$\mathbb{S}\mathscr{V}^{\perp\xi} \ni [\varepsilon, \varphi_0] \to [\iota_{\perp\xi}(\varepsilon), \alpha(\varphi_0)] \in \mathbb{S}\mathscr{V}$$

is an isomorphism of vector bundles such that $\iota_{\perp\xi}$ is a homomorphism of principal bundles and α is a homomorphism of $Spin_{p-1}$-representations with respect to the group homomorphism that is inclusion $Spin_{p-1} \subset Spin_p$. These properties mean $\mathbb{S}\mathscr{V}^{\perp\xi} \to \mathbb{S}\mathscr{V}$ is an isomorphism of associated bundles. It will be more convenient for us to consider this map going the other way, so we denote that direction by

$$\mathbb{S}\mathscr{V} \to \mathbb{S}\mathscr{V}^{\perp\xi} \;:\; \varphi \to \varphi^{\perp\xi} \;.$$

However, $\mathbb{S}\mathscr{V}^{\perp\xi}$ and $\mathbb{S}\mathscr{V}$ have more structure than even their associated bundle structure—they are also acted upon by the Clifford bundles $\mathbb{C}l(\mathscr{V}^{\perp\xi})$ and $\mathbb{C}l(\mathscr{V})$. The difference is purely that we are not allowed to act ξ on $\mathbb{S}\mathscr{V}^{\perp\xi}$ in the usual way, and we use the superscript ${}^{\perp\xi}$ to remind ourselves of this. It is with this in mind we make the following definition.

Definition 5.1.7. *The standard action of ξ on $\mathbb{S}\mathscr{V}$ will be called the **illegal action** and denoted by*

$$\xi \star \varphi\ , \qquad \varphi \in \mathbb{S}\mathscr{V}\ .$$

The special name and symbol for this will help us avoid confusion between this and the standard action of ξ on $\mathbb{S}\mathcal{M}$.

Part *4* of Theorem 5.1.5 shows that a spinor field on $\mathcal{M}$ may be constructed from a section of $\mathbb{S}\mathscr{V}^{\perp\xi}$ and a pair of horizontal spinor fields. But we want to construct one from a *vertical* spinor field and a pair of horizontal fields, so how can we do this? As pointed out above, $\mathbb{S}\mathscr{V}^{\perp\xi}$ and $\mathbb{S}\mathscr{V}$ are isomorphic as associated bundles, in a natural way. This isomorphism does not tell us how Clifford multiplication works—should elements of $\mathscr{V}^{\perp\xi} \subset \mathscr{V}$ act in the same way with respect to the isomorphism?
Consider the unique irreducible complex representation Δ_{p-1} of $\mathbb{C}l_{p-1}$ for p odd. How can we use this representation to construct one of $\mathbb{C}l_p$, which we know must act on the same underlying vector space?

Definition 5.1.8. *Let p be odd. Denote Clifford multiplication corresponding to the representation Δ_{p-1} by $\cdot_{p-1}$ and let $v_2, \dots, v_p$ be a frame for $\mathbb{R}^{p-1} \subset \mathbb{R}^p$, so that v_1 will be ξ when we apply our reasoning to bundles. The first representation of $\mathbb{C}l_p$ we shall define will have Clifford multiplication denoted by $\cdot_{p1}$ and is defined by the standard inclusion $\mathbb{C}l_{p-1} \subset \mathbb{C}l_p$ given by $\mathbb{R}^{p-1} \subset \mathbb{R}^p$:*

$$v_i \cdot_{p1} \stackrel{\text{def}}{=} v_i \cdot_{p-1} \qquad 2 \le i \le p$$

on Δ_{p-1}. This definition fixes the action of v_1 up to sign, and a calculation shows that

$$v_1 \cdot_{p1} = \begin{pmatrix} \pm i^{3p} & 0 \\ 0 & \mp i^{3p} \end{pmatrix}$$

is the only possibility, written with respect to the splitting of Δ_{p-1} under $\mathbb{C}l^0_{p-1}$. This gives us the representation $\Delta^{\pm}_p$ on the same underlying space Δ_{p-1}. A second representation of $\mathbb{C}l_p$ will have Clifford multiplication denoted by $\cdot_{p2}$ and is defined using the isomorphism

$$\mathbb{C}l_{p-1} \xrightarrow{\sim} \mathbb{C}l^0_p \subset \mathbb{C}l_p\ :\ v_i \to v_1 v_i \qquad 2 \le i \le p\ .$$

We have

$$v_i \cdot_{p2} \stackrel{\text{def}}{=} v_1 \cdot_{p2} v_i \cdot_{p-1} \qquad 2 \le i \le p$$

where $v_1 \cdot_{p2}$ is yet to be determined. A simple calculation shows that the action of v_1 must be the same as for the first representation:

$$v_1 \cdot_{p2} = v_1 \cdot_{p1} = \begin{pmatrix} \pm i^{3p} & 0 \\ 0 & \mp i^{3p} \end{pmatrix}$$

because $p-1$ is even. This describes $\Delta^{\pm}_p$ in a different way.

When p is odd, we have described two ways to extend Clifford multiplication on Δ_{p-1} to include the extra vector v_1, and these must result in the same representation $\Delta_p^{\pm}$. One may check that the map

$$\begin{pmatrix} I & 0 \\ 0 & v_1\cdot \end{pmatrix}$$

intertwines these two notations, where $v_1\cdot$ is the action with respect to either, since they are the same.

Suppose we are given a vertical field φ and horizontal fields χ and ψ. If $\varphi^{\perp\xi}$ is the image of φ under the natural isomorphism of associated bundles $\mathbb{S}\mathscr{V} \to \mathbb{S}\mathscr{V}^{\perp\xi}$, then $\varphi^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, \psi]$ is a spinor field on $\mathcal{M}$. Using $\cdot_{p1}$,

$$v_i \cdot \varphi = \left(v_i \cdot \varphi^{\perp\xi}\right)^{\perp\xi^{-1}}$$

for $i = 2, \ldots, p$ and so

$$(v_i \cdot \varphi^{\perp\xi}) \mathbin{\hat{\otimes}} [\chi, \psi] = (v_i \cdot \varphi)^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, \psi] \ .$$

Using $\cdot_{p2}$,

$$v_i \cdot \varphi = \xi \star \left(v_i \cdot \varphi^{\perp\xi}\right)^{\perp\xi^{-1}}$$

for $i = 2, \ldots, p$ and so

$$(v_i \cdot \varphi^{\perp\xi}) \mathbin{\hat{\otimes}} [\chi, \psi] = -(\xi \star v_i \cdot \varphi)^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, \psi] \ .$$

The illegal action of ξ is always

$$\xi \star \varphi = \pm i^{3p} \bar{\varphi} \ .$$

As a final point in this section, we have to understand how to lift a spinor field on the base space $\mathcal{B}$ of the submersion to a horizontal spinor field on the total space $\mathcal{M}$. We have already noted the existence of the map $\Xi^{Spin} : Spin(\mathscr{H}) \to Spin(\mathcal{B})$.

Definition 5.1.9. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion between Riemannian spin manifolds. The **basic lift** of a spinor field $\check{\chi}$ on $\mathcal{B}$ is the horizontal spinor field χ on $\mathcal{M}$ given in associated bundle notation by*

$$\begin{gathered} \mathbb{S}\mathcal{B} = Spin(\mathcal{B}) \times_{Spin_n} \Delta_n \to Spin(\mathscr{H}) \times_{Spin_n} \Delta_n = \mathbb{S}\mathscr{H} : \\ [\Xi^{Spin}(\varepsilon), \chi_0] \to [\varepsilon, \lambda\chi_0 \circ \pi] \ . \end{gathered}$$

The above definition is motivated by the way basic lifts of vector fields work; if

$$\check{X} = \sum_{i=1}^{n} X^i e_i = [e, X_0] = [\Xi(\lambda^{-1} e), X_0]$$

is a vector field on $\mathcal{B}$ then the basic lift X is

$$X = \sum_{i=1}^{n} (X^i \circ \pi) e_i = \sum_{i=1}^{n} \lambda (X^i \circ \pi)(\lambda^{-1} e_i) = [\lambda^{-1} e, \lambda X_0 \circ \pi]$$

and the factor of λ will help our spinor equations to more closely resemble our vector ones (although we could define it without the λ if we wanted to).

Proposition 5.1.10. *When χ is the basic lift of the spinor field $\check{\chi}$ and X is the basic lift of the vector field $\check{X}$, the lift of $\check{X} \cdot \check{\chi}$ is $\lambda^{-1} X \cdot \chi$.*

Proof. Simple calculation. □

5.2 O'Neill formulae for derivatives

We begin this section with a well-known isomorphism, which for simplicity we write in n dimensions although it of course holds in p or $p+n$ dimensions too.

Lemma 5.2.1. *Let $(e_1 \dots, e_n)$ be the standard frame of $\mathbb{R}^n$. Then*

$$\mathfrak{so}_n = span_{\mathbb{R}}\{e_i \wedge e_j\}$$

where our convention is always to act on the first argument first, i.e.

$$e_i \wedge e_j(e_k) = \frac{1}{2}\delta_{ik} e_j - \frac{1}{2}\delta_{jk} e_i \ .$$

Some authors will use minus this, and often without the factors of $1/2$. We can also write

$$\mathfrak{spin}_n = span_{\mathbb{R}}\{e_i e_j\}$$

where the product is Clifford product. Then

$$\mathfrak{so}_n \xrightarrow{\sim} \mathfrak{spin}_n \ : \ e_i \wedge e_j \to \frac{1}{4} e_i e_j$$

is an isomorphism.

Proof. This is clear from

$$[e_i \wedge e_j, e_j \wedge e_k] = -\frac{1}{2} e_i \wedge e_k \ , \quad [e_i e_j, e_j e_k] = -2 e_i e_k \ .$$

□

Now suppose Φ is a spinor field on the total space $\mathcal{M}$. We can represent it as $[\varepsilon, \Phi_0]$, a section of $Spin(\mathcal{M}, \mathscr{V}) \times_{Spin_p \times_{\mathbb{Z}_2} Spin_n} \Delta_{p+n}$. Denote the spin structure double covering map by $\eta : Spin(\mathcal{M}, \mathscr{V}) \to SO(\mathcal{M}, \mathscr{V})$ and suppose $\eta(\varepsilon) = (v, \lambda^{-1} e)$. The spin connection is the pullback of the Levi-Civita connection: $\omega^{Spin} = \eta^* \omega$, and the connection coefficients of ω are defined by

$$\Gamma_Z^{(v,\lambda^{-1}e)} \stackrel{\text{def}}{=} \big((v, \lambda^{-1} e)^* \omega\big)(Z) = (v, \lambda^{-1} e)^* \big(\omega((v, \lambda^{-1} e)_* Z)\big)$$

for any vector field Z on $\mathcal{M}$. The covariant derivative of Φ may be written as

$$\begin{aligned}
\nabla_Z \Phi &= [\varepsilon, Z\Phi_0 + \omega^{Spin}(\varepsilon_* Z)\Phi_0] \\
&= [\varepsilon, Z\Phi_0 + (\eta^*\omega)(\eta_*^{-1}(v, \lambda^{-1}e)_* Z)\Phi_0] \\
&= [\varepsilon, Z\Phi_0 + \eta_*^{-1}\big(\omega((v, \lambda^{-1}e)_* Z)\big)\Phi_0] \\
&= [\varepsilon, Z\Phi_0] + [(v, \lambda^{-1}e), \eta_*^{-1}\big(\omega((v, \lambda^{-1}e)_* Z)\big)] \cdot [\varepsilon, \Phi_0]
\end{aligned}$$

where we've used the symbol η also for the standard double covering $Spin_{p+n} \to SO_{p+n}$, so that $\eta_*^{-1} : \mathfrak{so}_{p+n} \to \mathfrak{spin}_{p+n}$ is exactly our isomorphism explained in Lemma 5.2.1. Since by definition

$$[(v, \lambda^{-1}e), \eta_*^{-1}\big(\omega((v, \lambda^{-1}e)_* Z)\big)] = \Gamma_Z^{(v,\lambda^{-1}e)} \ ,$$

we arrive at the following well-known fact.

Lemma 5.2.2. *With respect to the local frame $(v, \lambda^{-1}e)$, we can write*

$$\nabla_Z \Phi = Z\Phi + \frac{1}{4}\Gamma_Z^{(v,\lambda^{-1}e)} \cdot \Phi$$

for any spinor field Φ and vector field Z on $\mathcal{M}$.

The main result of this section is the application of Lemma 5.2.2 to Proposition 3.3.3 by using the formulae of Theorem 5.1.5. Before we state this we need one more definition, which will help us in the hardest case when both $\mathscr{V}$ and $\mathscr{H}$ have odd dimension.

Definition 5.2.3. *In 4 of Theorem 5.1.5 it was necessary to work with $\mathscr{V}^{\perp\xi}$ instead of $\mathscr{V}$, and we shall need to do this again for the corresponding part of the theorem that follows. We define a covariant derivative operator $\hat{\nabla}^{\perp\xi}$ on $\mathscr{V}^{\perp\xi}$ by projecting $\hat{\nabla}$:*

$$(\hat{\nabla}^{\perp\xi})_Z U \stackrel{\text{def}}{=} (\hat{\nabla}_Z U)^{\perp\xi}$$

for any vector field Z on $\mathcal{M}$ and section U of $\mathscr{V}^{\perp\xi}$.

Theorem 5.2.4. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim\mathscr{V}, \dim\mathscr{H} > 1$. Let φ be a vertical spinor field and let χ and ψ be horizontal spinor fields on $\mathcal{M}$, as in the notation of Theorem 5.1.5, and consider the spinor field on $\mathcal{M}$ constructed from these. Then for X a horizontal and U a vertical vector field, the covariant derivatives are given by*

1. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both even:***

$$\nabla_X(\varphi \mathbin{\hat{\otimes}} \chi) = (\hat{\nabla}_X\varphi) \mathbin{\hat{\otimes}} \chi + \varphi \mathbin{\hat{\otimes}} (\nabla_X^{\mathscr{H}}\chi) + \frac{1}{4}A_X^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ - \frac{1}{2\lambda}\varphi \mathbin{\hat{\otimes}} X \cdot hgrad^g\lambda \cdot \chi - \frac{X(\lambda)}{2\lambda}\varphi \mathbin{\hat{\otimes}} \chi \ ,$$

$$\nabla_U(\varphi \mathbin{\hat{\otimes}} \chi) = (\hat{\nabla}_U\varphi) \mathbin{\hat{\otimes}} \chi + \varphi \mathbin{\hat{\otimes}} (\nabla_U^{\mathscr{H}}\chi) + \frac{1}{4}\varphi \mathbin{\hat{\otimes}} A^g U \cdot \chi \\ + \frac{1}{4}T_U^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \ .$$

2. $\dim\mathscr{V}$ ***even*** *and* $\dim\mathscr{H}$ ***odd:*** *The formulae are the same as in the case above.*

3. $\dim\mathscr{V}$ ***odd*** *and* $\dim\mathscr{H}$ ***even:*** *The formulae are the same as in the case above, except with φ and χ swapped as in 3 of Theorem 5.1.5.*

4. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both odd:*** *In this case $\varphi^{\perp\xi}$ is not a vertical spinor field but rather a*

section of $\mathbb{S}\mathscr{V}^{\perp\xi}$, and ξ is a unit vertical vector field defined locally as in Theorem 5.1.5.

$$\begin{aligned}\nabla_X\big(\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big]\big) = {} & (\hat{\nabla}^{\perp\xi}_X\varphi^{\perp\xi})\,\hat{\otimes}\,\big[\chi,\psi\big] + \varphi^{\perp\xi}\,\hat{\otimes}\,\big[\nabla^{\mathscr{H}}_X\chi,\nabla^{\mathscr{H}}_X\psi\big]\\ & + \frac{1}{4}A^g_X\cdot\big(\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big]\big)\\ & - \frac{1}{2\lambda}\varphi^{\perp\xi}\,\hat{\otimes}\,\big[X\cdot hgrad^g\lambda\cdot\chi, X\cdot hgrad^g\lambda\cdot\psi\big]\\ & - \frac{X(\lambda)}{2\lambda}\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big] - \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi})\,\hat{\otimes}\,\big[-\chi,\psi\big]\ ,\end{aligned}$$

$$\begin{aligned}\nabla_U\big(\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big]\big) = {} & (\hat{\nabla}^{\perp\xi}_U\varphi^{\perp\xi})\,\hat{\otimes}\,\big[\chi,\psi\big] + \varphi^{\perp\xi}\,\hat{\otimes}\,\big[\nabla^{\mathscr{H}}_U\chi,\nabla^{\mathscr{H}}_U\psi\big]\\ & + \frac{1}{4}\varphi^{\perp\xi}\,\hat{\otimes}\,\big[A^gU\cdot\chi, A^gU\cdot\psi\big]\\ & + \frac{1}{4}T^g_U\cdot\big(\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big]\big)\\ & - \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_U\xi\cdot\varphi^{\perp\xi})\,\hat{\otimes}\,\big[-\chi,\psi\big]\ .\end{aligned}$$

Proof. To prove *1*, begin with

$$\nabla_X(\varphi\,\hat{\otimes}\,\chi) = X(\varphi\,\hat{\otimes}\,\chi) + \frac{1}{4}\Gamma^{(v,\lambda^{-1}e)}_X\cdot(\varphi\,\hat{\otimes}\,\chi)\ .$$

We can express $\varphi\,\hat{\otimes}\,\chi$ using the associated bundle notation as $[\varepsilon,\varphi_0\,\hat{\otimes}\,\chi_0]$, where φ_0 is a Δ_p-valued function on $\mathcal{M}$, χ_0 is a Δ_n-valued one, and $\eta(\varepsilon) = (v,\lambda^{-1}e)$. The Leibniz rule is now clear:

$$\begin{aligned}X[\varepsilon,\varphi_0\,\hat{\otimes}\,\chi_0] &= [\varepsilon, X(\varphi_0\,\hat{\otimes}\,\chi_0)]\\ &= [\varepsilon,(X\varphi_0)\,\hat{\otimes}\,\chi_0 + \varphi_0\,\hat{\otimes}\,(X\chi_0)]\\ &= X[\varepsilon,\varphi_0]\,\hat{\otimes}\,[\varepsilon,\chi_0] + [\varepsilon,\varphi_0]\,\hat{\otimes}\,X[\varepsilon,\chi_0]\ .\end{aligned}$$

Proposition 3.3.3 tells us that

$$\begin{aligned}\Gamma^{(v,\lambda^{-1}e)}_X\cdot(\varphi\,\hat{\otimes}\,\chi) = {} & \varphi\,\hat{\otimes}\,(\pi^*\check{\Gamma}^e_X\cdot\chi) - \frac{2}{\lambda}\varphi\,\hat{\otimes}\,(X\wedge hgrad^g\lambda\cdot\chi)\\ & + A^g_X\cdot(\varphi\,\hat{\otimes}\,\chi) + (\hat{\Gamma}^v_X\cdot\varphi)\,\hat{\otimes}\,\chi\end{aligned}$$

from which the first equation is now clear, remembering that

$$\nabla^{\mathscr{H}} = d^{\lambda^{-1}e} + \pi^*\check{\Gamma}^e\ .$$

The second equation is similar, but we need that if ε^h is in $Spin(\mathscr{H})$ and maps to $\lambda^{-1}e$ under $Spin(\mathscr{H})\to SO(\mathscr{H})$, then

$$d^{\varepsilon^h}\chi(U) = d^{\varepsilon^h}\chi(U) + \frac{1}{4}\Gamma^{\lambda^{-1}e}_{\mathscr{H}U}\cdot\chi = \nabla^{\mathscr{H}}_U\chi$$

because $\Gamma^{\lambda^{-1}e}_{\mathscr{H}U} = (\pi^*\check{\Gamma}^e)_U = 0$. This is enough to prove *1*, and parts *2* and *3* are very similar. Part *4* is a little harder. Firstly, in the formula we have for $\Gamma^{(v,\lambda^{-1}e)}_X$, the terms $\pi^*\check{\Gamma}^e_X$ and $X\wedge hgrad^g\lambda$ are horizontal and so act on $\varphi^{\perp\xi}\,\hat{\otimes}\,\big[\chi,\psi\big]$ in the same way as before. The term

A^g_X may indeed have a part involving ξ but since we're not going to try to distribute it over the tensor product we can leave it alone. The problem term is $\hat{\Gamma}^v_X$, which may contain a ξ-part which will act by the extra formula as explained in Theorem 5.1.5. Since ξ is a unit vertical vector field, we can choose the local vertical frame v so that $\xi = v_1$. If we write $\hat{\Gamma}^v_X = \sum_{i,j=1}^p (\hat{\Gamma}^v_X)^{ij} v_i \wedge v_j$ then

$$\begin{aligned}\hat{\Gamma}^v_X &= \sum_{j=2}^p (\hat{\Gamma}^v_X)^{1j} \xi \wedge v_j + \sum_{i=2}^p (\hat{\Gamma}^v_X)^{i1} v_i \wedge \xi + \sum_{i,j=2}^p (\hat{\Gamma}^v_X)^{ij} v_i \wedge v_j \\ &= 2\sum_{j=2}^p (\hat{\Gamma}^v_X)^{1j} \xi \wedge v_j + \hat{\Gamma}^{v\perp\xi}_X\end{aligned}$$

where we've put $\hat{\Gamma}^{v\perp\xi}_X \overset{\text{def}}{=} \sum_{i,j=2}^p (\hat{\Gamma}^v_X)^{ij} v_i \wedge v_j$. Now

$$\hat{\Gamma}^v_X \xi = \sum_{j=2}^p (\hat{\Gamma}^v_X)^{1j} v_j$$

so

$$\hat{\Gamma}^v_X = 2\xi \wedge \hat{\Gamma}^v_X \xi + \hat{\Gamma}^{v\perp\xi}_X \ .$$

Since ξ is a member of the frame v, $d^v \xi = 0$ and

$$\hat{\Gamma}^v \xi = d^v \xi + \hat{\Gamma}^v \xi = \hat{\nabla}\xi \ .$$

Finally we remark that since $\hat{\nabla}^{\perp\xi}$ is defined as the projection of $\hat{\nabla}$, its coefficients with respect to v are precisely $\hat{\Gamma}^{v\perp\xi}$. For $\varepsilon^v \in Spin(\mathscr{V})$ we have

$$\begin{aligned}&(d^{\varepsilon^v} \varphi^{\perp\xi})(X) \,\hat{\otimes}\, [\chi, \psi] + \frac{1}{4}\hat{\Gamma}^v_X \cdot (\varphi^{\perp\xi} \,\hat{\otimes}\, [\chi,\psi]) \\ &= (d^{\varepsilon^v} \varphi^{\perp\xi})(X) \,\hat{\otimes}\, [\chi, \psi] + \frac{1}{4}(\hat{\Gamma}^{v\perp\xi}_X \cdot \varphi^{\perp\xi}) \,\hat{\otimes}\, [\chi,\psi] + \frac{1}{2}\xi \cdot [(\hat{\nabla}_X \xi \cdot \varphi^{\perp\xi}) \,\hat{\otimes}\, [\psi, \chi]] \\ &= (\hat{\nabla}^{\perp\xi}_X \varphi^{\perp\xi}) \,\hat{\otimes}\, [\chi,\psi] - \frac{1}{2} i^{\dim \mathcal{M}} (\hat{\nabla}_X \xi \cdot \varphi^{\perp\xi}) \,\hat{\otimes}\, [-\chi, \psi] \ .\end{aligned}$$

The second formula is similar. □

Remark 5.2.5. *We can define pseudo-conformal submersions between pseudo-Riemannian spin manifolds and perform similar calculations to derive O'Neill formulae for the covariant derivatives in this setting. Note that non-degeneracy of the metric on $\mathcal{B}$ implies non-degeneracy of the metric on the horizontal distribution $\mathscr{H}$, from which it follows that the vertical distribution $\mathscr{V}$ must have a non-degenerate metric. We do not take this further in this thesis.*

Whilst we are satisfied with parts *1-3* of Theorem 5.2.4, the fourth part is not what we would like. Instead of the term $\hat{\nabla}^{\perp\xi}_X \varphi^{\perp\xi}$ we would prefer a term with $(\hat{\nabla}_X \varphi)^{\perp\xi}$; we want to use the derivative of a vertical spinor field rather than a section of $\mathbb{S}\mathscr{V}^{\perp\xi}$. We will now try to modify the formula of part *4*.

Lemma 5.2.6. *With respect to the natural isomorphism of vector bundles associated to $Spin(\mathscr{V}^{\perp\xi})$*

$$\mathbb{S}\mathscr{V}^{\perp\xi} \cong \mathbb{S}\mathscr{V} \ ,$$

if we use Clifford multiplication $\cdot_{p1}$ *then*

$$(\hat{\nabla}_W\varphi)^{\perp\xi} = \hat{\nabla}_W^{\perp\xi}\varphi^{\perp\xi} \mp \frac{1}{2}i^{3p}\hat{\nabla}_W\xi\cdot\overline{\varphi^{\perp\xi}}$$

and if we use Clifford multiplication $\cdot_{p2}$ *then*

$$(\hat{\nabla}_W\varphi)^{\perp\xi} = \hat{\nabla}_W^{\perp\xi}\varphi^{\perp\xi} - \frac{1}{2}\hat{\nabla}_W\xi\cdot\varphi^{\perp\xi}$$

where $\varphi^{\perp\xi}$ *is a section of* $\mathbb{S}\mathscr{V}^{\perp\xi}\cong\mathbb{S}\mathscr{V}$ *and* W *is any vector field on* $\mathcal{M}$.

Proof. If U is a section of $\mathscr{V}^{\perp\xi}$,

$$\hat{\nabla}_W U = \hat{\nabla}_W^{\perp\xi}U + g(\hat{\nabla}_W U,\xi)\xi\ ,$$

and $g(\hat{\nabla}_W U,\xi) = -g(U,\hat{\nabla}_W\xi)$ so

$$\hat{\nabla}_W = \hat{\nabla}_W^{\perp\xi} - 2\hat{\nabla}_W\xi\wedge\xi\ .$$

We will prove the $\cdot_{p2}$ formula in more detail, and then the first is very similar. In terms of the isomorphism $\perp\xi$, the Clifford multiplication $\cdot_{p2}$ of Definition 5.1.8 is written as

$$\xi\star\varphi = \xi\star(\varphi^{\perp\xi})^{\perp\xi^{-1}} = \pm i^{3p}(\overline{\varphi^{\perp\xi}})^{\perp\xi^{-1}}\ ,$$

and

$$U\cdot\varphi = U\cdot(\varphi^{\perp\xi})^{\perp\xi^{-1}} = \xi\star(U\cdot\varphi^{\perp\xi})^{\perp\xi^{-1}}\ .$$

Using Lemma 5.2.2,

$$\begin{aligned}
(\hat{\nabla}_W^{\perp\xi}\varphi^{\perp\xi})^{\perp\xi^{-1}} &= \hat{\nabla}_W\varphi - \frac{1}{2}(\xi\wedge\hat{\nabla}_W\xi)\cdot\varphi\\
&= \hat{\nabla}_W\varphi - \frac{1}{2}\xi\star\hat{\nabla}_W\xi\cdot\varphi\\
&= \hat{\nabla}_W\varphi - \frac{1}{2}\xi\star\left(\xi\star(\hat{\nabla}_W\xi\cdot\varphi^{\perp\xi})^{\perp\xi^{-1}}\right)\\
&= \hat{\nabla}_W\varphi + \frac{1}{2}(\hat{\nabla}_W\xi\cdot\varphi^{\perp\xi})^{\perp\xi^{-1}}\ ,
\end{aligned}$$

and applying $\perp\xi$ to both sides yields the formula. $\square$

Lemma 5.2.7. *In the formula for* $\nabla_X\big(\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[\chi,\psi]\big)$ *in part 4 of Theorem 5.2.4, the following terms can be rewritten as*

$$\begin{aligned}
(\hat{\nabla}_X^{\perp\xi}\varphi^{\perp\xi})\mathbin{\hat{\otimes}}\big[\chi,\psi\big] &- \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat{\otimes}}\big[-\chi,\psi\big]\\
&= (\hat{\nabla}_X\varphi)^{\perp\xi}\mathbin{\hat{\otimes}}\big[\chi,\psi\big] + \frac{1}{2}(\mp i^{3p}\hat{\nabla}_X\xi\cdot\overline{\varphi^{\perp\xi}} + i^{\dim\mathcal{M}}\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat{\otimes}}\big[\chi,0\big]\\
&\qquad + \frac{1}{2}(\mp i^{3p}\hat{\nabla}_X\xi\cdot\overline{\varphi^{\perp\xi}} - i^{\dim\mathcal{M}}\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat{\otimes}}\big[0,\psi\big]
\end{aligned}$$

with respect to $\cdot_{p1}$*, or*

$$(\hat{\nabla}_X^{\perp\xi}\varphi^{\perp\xi}) \hat{\otimes} \big[\chi,\psi\big] - \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[-\chi,\psi\big]$$
$$= (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[\chi,\psi\big] + \frac{1}{2}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi} + i^{\dim\mathcal{M}}\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[\chi,0\big]$$
$$+ \frac{1}{2}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi} - i^{\dim\mathcal{M}}\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[0,\psi\big]$$
$$= \begin{cases} (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[\chi,\psi\big] + (\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[\chi,0\big] & \dim\mathcal{M} = 0 \bmod 4 \\ \\ (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[\chi,\psi\big] + (\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[0,\psi\big] & \dim\mathcal{M} = 2 \bmod 4 \end{cases}$$

with respect to $\cdot_{p2}$*. Similarly for* $\nabla_U\big(\varphi^{\perp\xi} \hat{\otimes} \big[\chi,\psi\big]\big)$.

Proof. Follows from Lemma 5.2.6. □

Consider what happens when either $\chi = 0$ or $\psi = 0$, which we will put when we have a vertical spinor field and only one spinor field on the base space of the submersion. We want to know when one of the conditions

$$(\hat{\nabla}_X^{\perp\xi}\varphi^{\perp\xi}) \hat{\otimes} \big[\chi,0\big] - \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[-\chi,0\big] = (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[\chi,0\big]$$

or

$$(\hat{\nabla}_X^{\perp\xi}\varphi^{\perp\xi}) \hat{\otimes} \big[0,\psi\big] - \frac{1}{2}i^{\dim\mathcal{M}}(\hat{\nabla}_X\xi\cdot\varphi^{\perp\xi}) \hat{\otimes} \big[0,\psi\big] = (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[0,\psi\big]$$

is satisfied. If we use $\cdot_{p1}$ then we require either

$$-i^{\dim\mathcal{M}}\varphi^{\perp\xi} = \mp i^{3p}\overline{\varphi^{\perp\xi}} = -(\xi\star\varphi)^{\perp\xi}$$

or

$$i^{\dim\mathcal{M}}\varphi^{\perp\xi} = \mp i^{3p}\overline{\varphi^{\perp\xi}} = -(\xi\star\varphi)^{\perp\xi}$$

and neither of these is satisfied. This spells the end of the line for the multiplication $\cdot_{p1}$ of Definition 5.1.8. If we use $\cdot_{p2}$ and if $\dim\mathcal{M} = 0 \bmod 4$ then $\chi = 0$ implies the second condition holds (but not the first), and similarly if $\dim\mathcal{M} = 2 \bmod 4$ then $\psi = 0$ implies the first condition holds (but not the second). In other words, the rule is that *when we take the basic lift of a spinor field when* $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ *are both odd, we always define the lift to be of negative chirality.* From now on we will only use $\cdot_{p2}$.

Theorem 5.2.8. *We can modify part 4 of Theorem 5.2.4.*
$\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both odd****,* $\dim\mathcal{M} = 2 \bmod 4$*: Now* φ *is a vertical spinor field and* χ *is horizontal, and Clifford multiplication has been extended from* $\mathbb{S}\mathscr{V}^{\perp\xi}$ *by* $\cdot_{p2}$ *to produce* $\mathbb{S}\mathscr{V}$*.*

$$\nabla_X\big(\varphi^{\perp\xi} \hat{\otimes} \big[\chi,0\big]\big) = (\hat{\nabla}_X\varphi)^{\perp\xi} \hat{\otimes} \big[\chi,0\big] + \varphi^{\perp\xi} \hat{\otimes} \big[\nabla_X^{\mathscr{H}}\chi,0\big]$$
$$+ \frac{1}{4}A_X^g\cdot\big(\varphi^{\perp\xi} \hat{\otimes} \big[\chi,0\big]\big)$$
$$- \frac{1}{2\lambda}\varphi^{\perp\xi} \hat{\otimes} \big[X\cdot hgrad^g\lambda\cdot\chi,0\big]$$
$$- \frac{X(\lambda)}{2\lambda}\varphi^{\perp\xi} \hat{\otimes} \big[\chi,0\big]$$

$$\begin{aligned}\nabla_U\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,0\big]\big) &= (\hat\nabla_U\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,0\big]+\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\nabla^{\mathscr{H}}_U\chi,0\big]\\ &\quad+\frac{1}{4}\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[A^gU\cdot\chi,0\big]\\ &\quad+\frac{1}{4}T^g_U\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,0\big]\big)\end{aligned}$$

When $\dim\mathcal{M}=0 \bmod 4$, *the equations are the same except we use* $\big[0,\psi\big]$ *throughout instead.*

Proof. Apply Lemma 5.2.7. □

This makes the covariant derivative formulae in the case of $\dim\mathscr{V}$ and $\dim\mathscr{H}$ both odd the same as the other cases except with χ replaced by $\big[\chi,0\big]$ and with the isomorphism ${}^{\perp\xi}$ involved.

Remark 5.2.9. *The modification of part 4 of Theorem 5.2.4 through Lemma 5.2.6 and Lemma 5.2.7 amounts to the following calculation.*

$$\begin{aligned}\hat\Gamma^v_X\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]\big) &= \hat\Gamma^{v\perp\xi}_X\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]\big)+\frac{1}{2}\xi\wedge\hat\nabla_X\xi\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]\big)\\ &= (\hat\Gamma^{v\perp\xi}_X\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[\chi,\psi\big]-\frac{1}{2}i^{\dim\mathcal{M}}(\hat\nabla_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[-\chi,\psi\big]\end{aligned}$$

and

$$\begin{aligned}\hat\Gamma^{v\perp\xi}_X\cdot\varphi^{\perp\xi} &= (\hat\Gamma^{v\perp\xi}_X\cdot\varphi)^{\perp\xi}\\ &= (\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}-\frac{1}{2}(\xi\star\hat\nabla_X\xi\cdot\varphi)^{\perp\xi}\\ &= (\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}+\frac{1}{2}\hat\nabla_X\xi\cdot\varphi^{\perp\xi}\ .\end{aligned}$$

Thus

$$\begin{aligned}\hat\Gamma^v_X\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]\big) &= (\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]+\frac{1}{2}(\hat\nabla_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[\chi,\psi\big]\\ &\quad-i^{\dim\mathcal{M}}\frac{1}{2}(\hat\nabla_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[-\chi,\psi\big]\\ &= \begin{cases}(\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]+(\hat\nabla_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[\chi,0\big] & \dim\mathcal{M}=0 \bmod 4\\ \\ (\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,\psi\big]+(\hat\nabla_X\xi\cdot\varphi^{\perp\xi})\mathbin{\hat\otimes}\big[0,\psi\big] & \dim\mathcal{M}=2 \bmod 4\end{cases}\end{aligned}$$

and so if $\chi=0$ *and* $\dim\mathcal{M}=0 \bmod 4$ *then*

$$\hat\Gamma^v_X\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[0,\psi\big]\big)=(\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[0,\psi\big]$$

and if $\psi=0$ *and* $\dim\mathcal{M}=2 \bmod 4$ *then*

$$\hat\Gamma^v_X\cdot\big(\varphi^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,0\big]\big)=(\hat\Gamma^v_X\cdot\varphi)^{\perp\xi}\mathbin{\hat\otimes}\big[\chi,0\big]\ .$$

Note that this calculation does not work when we try to use $\cdot_{p1}$ *instead of* $\cdot_{p2}$.

Now we can return to the general case, with no assumptions on the parity of the dimensions.

Theorem 5.2.10. *Let* $\pi:\mathcal{M}\to\mathcal{B}$ *be a conformal submersion of Riemannian spin manifolds where both base and fibre have dimension greater than* 1. *With respect to the realisations of* *the spinor bundles as in Theorem 5.1.5, where φ is vertical and χ is horizontal, the covariant derivative satisfies a 'Leibniz-type rule'*

$$\nabla_W\big(\varphi \mathbin{\hat{\otimes}} \chi\big) = (\hat{\nabla}_W\varphi) \mathbin{\hat{\otimes}}\chi + \varphi \mathbin{\hat{\otimes}} (\nabla^{\mathscr{H}}_W\chi) \qquad \forall W \in \mathfrak{X}(\mathcal{M})$$

if and only if $\mathscr{H}$ is integrable, π has totally geodesic fibres and λ is constant. In the case of $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ *both odd and* $\dim\mathcal{M} = 2 \bmod 4$ *the formula is*

$$\nabla_W\big(\varphi^{\perp\xi} \mathbin{\hat{\otimes}} \big[\chi, 0\big]\big) = (\hat{\nabla}_W\varphi)^{\perp\xi} \mathbin{\hat{\otimes}} \big[\chi, 0\big] + \varphi^{\perp\xi} \mathbin{\hat{\otimes}} \big[\nabla^{\mathscr{H}}_W\chi, 0\big]$$

and is similar for $\dim\mathcal{M} = 0 \bmod 4$.

Proof. Consider the formulae presented in Theorem 5.2.4. Recall

$$A^g_W = A^{\lambda^{-2}g}_W - \frac{2}{\lambda} W \wedge vgrad^g\lambda$$

for any vector field W. Using the first formula, we see that the Leibniz rule works for a horizontal vector field W if and only if

$$\frac{1}{4}A^{\lambda^{-2}g}_W \cdot (\varphi \mathbin{\hat{\otimes}} \chi) - \frac{1}{2\lambda} W \wedge grad^g\lambda \cdot (\varphi \mathbin{\hat{\otimes}} \chi) = 0 \ .$$

Since for Y horizontal $A^{\lambda^{-2}g}_W Y$ is skew in W and Y, $A^{\lambda^{-2}g}_W$ cannot have horizontal part proportional to W. It is a rank two tensor field which swaps the horizontal and vertical distributions, so its horizontal part has rank one. The Clifford action of $A^{\lambda^{-2}g}_W$ on χ therefore must be distinct from that of W on χ, so the Leibniz rule holds for horizontal W if and only if

$$A^{\lambda^{-2}g}_W \cdot (\varphi \mathbin{\hat{\otimes}} \chi) = 0 \ , \quad W \wedge grad^g\lambda \cdot (\varphi \mathbin{\hat{\otimes}} \chi) = 0 \ ,$$

which holds if and only if $A^{\lambda^{-2}g}_W = 0$ and $grad^g\lambda = 0$. From the second formula using a vertical vector field W it immediately follows that the Leibniz rule works if and only if $T^{\lambda^{-2}g}_W = T^g_W = 0$. Thus, it works for all $W \in \mathfrak{X}(\mathcal{M})$ if and only if $A^{\lambda^{-2}g} = 0$, so $\mathscr{H}$ is integrable, and $T^{\lambda^{-2}g} = 0$, so π has totally geodesic fibres, and $grad^g\lambda = 0$. The same reasoning works for the case of $\dim\mathscr{V}$ and $\dim\mathscr{H}$ both odd using Theorem 5.2.8. □

In the formulae of Theorems 5.2.4 and 5.2.8, the horizontal spinor field χ (and ψ) is not assumed to be *basic* in the sense of Definition 5.1.9. The following proposition may be used to modify the formulae in that case.

Proposition 5.2.11. *When χ is the basic lift of $\check{\chi}$,*

1. $\nabla^{\mathscr{H}}_X\chi$ *is the basic lift of* $\check{\nabla}_{\check{X}}\check{\chi}$ *plus* $\frac{X(\lambda)}{\lambda}\chi$,
2. $\nabla^{\mathscr{H}}_U\chi = \frac{U(\lambda)}{\lambda}\chi$.

The covariant derivative formulae can be modified accordingly.

The formulae of Theorem 5.2.4 may be used to aid our understanding of the following situation. Suppose $\mathcal{M}$ is a simply connected Riemannian manifold of dimension eight. There are inclusions

$$1 \subset Sp_1 \times Sp_1 \subset Sp_2 \subset SU_4 \subset Spin_7 \subset SO_8 \ .$$

These are some of the possible holonomy groups of $\mathcal{M}$ and each corresponds to a possible dimension of the space of parallel spinor fields on $\mathcal{M}$ (if $\mathcal{M}$ is not spin, its holonomy is generic or trivial). Wang's Theorem 1.0.1 tells us that the above groups SO_8, $Spin_7$, SU_4, Sp_2, $Sp_1 \times Sp_1$ and 1 correspond to the space of parallel spinor fields having dimension 0, 1, 2, 3, 4 and more. Suppose $\mathcal{M}$ has holonomy group $Sp_1 \times Sp_1$, so that $\mathcal{M}$ is a product of simply connected Calabi-Yau 4-manifolds (noting that $Sp_1 = SU_2$), by the de Rham Theorem. We shall write this as $\mathcal{M} = \mathcal{B} \times \mathcal{F}$. Since $\mathcal{B}$ and $\mathcal{F}$ are non-flat Calabi-Yau, they each admit precisely two independent parallel spinor fields (up to scale), and we call these φ_1, φ_2 and χ_1, χ_2 respectively. The projection $\mathcal{M} \to \mathcal{B}$ is a Riemannian submersion with $T = 0$ and $A = 0$. The spinor bundles $\mathbb{S}\mathcal{B}$ and $\mathbb{S}\mathcal{F}$ each have rank four and by Lemma 5.1.5 we know that $\mathbb{S}\mathcal{M} = \mathbb{S}\mathcal{B} \hat{\otimes} \mathbb{S}\mathcal{F}$. Theorem 5.2.4 (along with Proposition 5.2.11) gives

$$\nabla(\varphi_i \hat{\otimes} \chi_j) = (\hat{\nabla}\varphi_i) \hat{\otimes} \chi_j + \varphi_i \hat{\otimes} \check{\nabla}\chi_j = 0$$

for $i = 1, 2$ and $j = 1, 2$. Thus, we have reproduced a four-dimensional space of parallel spinor fields on $\mathcal{M}$ in terms of those on its factors $\mathcal{B}$ and $\mathcal{F}$. We can also see that there cannot be any more parallel spinor fields on $\mathcal{M}$ because the above expression does not vanish unless both φ_i and χ_j are parallel.

5.3 One-dimensional fibres

So far we have assumed $\dim \mathscr{V}, \dim \mathscr{H} > 1$ because otherwise the situation is simpler and well-known. A slightly simpler construction can be found in the paper of Bär, Gauduchon and Moroianu [BGM05]. In this section we will give a short account of the theory when $\dim \mathscr{V} = 1$ and $\dim \mathscr{H} = n > 1$. The case when both dimensions are equal to 1 is trivial.

As explained in Definition 5.1.8, if n is even then $\Delta^{\pm}_{n+1}$ has the same underlying space as Δ_n with the single vertical unit vector v_1 acting by

$$v_1 \cdot \chi = \pm i^{3n} \bar{\chi}$$

and the n horizontal vectors $e_1, \ldots, e_n$ by

$$e_j \cdot_{n+1} \chi = v_1 \cdot e_j \cdot_n \chi = \mp i^{3n} e_j \cdot_n \bar{\chi}$$

where the total frame $v_1, e_1, \ldots, e_n$ is of positive orientation. The corresponding statements can be made about the complex spinor bundles $\mathbb{S}\mathscr{H}$ and $\mathbb{S}\mathcal{M}$.

If n is odd, then Δ_{n+1} has the same underlying space as $\Delta^+_n + \Delta^+_n$ with the single vertical unit vector v_1 acting by

$$v_1 \cdot \big[\chi, \psi\big] = i^{3n+1}\big[-\psi, \chi\big]$$

and the n horizontal vectors $e_1, \ldots, e_n$ by

$$e_j \cdot_{n+1} \big[\chi, \psi\big] = \big[e_j \cdot_n \psi, e_j \cdot_n \chi\big]$$

where the total frame $v_1, e_1, \ldots, e_n$ is of positive orientation. The factor of i^{3n+1} is found by looking at the actions of the complex volume elements and is explained in more detail in Lemma 4.5.1. The corresponding statements can be made about the complex spinor bundles $\mathbb{S}\mathscr{H} + \mathbb{S}\mathscr{H}$

and $\mathbb{S}\mathcal{M}$.
Proposition 3.3.3 remains true as our derivation of the adapted connection coefficients does not depend on the fibre dimension. Definition 5.1.9 of the basic lift of a spinor field remains valid. Using these, we can find the covariant derivative of a spinor field on $\mathcal{M}$. We will omit the calculations as they are much simpler than those for high-dimensional fibres.

Theorem 5.3.1. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim \mathscr{V} = 1$ and $\dim \mathscr{H} > 1$. Let χ and ψ be horizontal spinor fields on $\mathcal{M}$, as in the notation above. Then for X a horizontal and U a vertical vector field, the covariant derivatives are given by*

1. $\dim \mathscr{H}$ ***even:***

$$\nabla_X \chi = \nabla^{\mathscr{H}}_X \chi + \frac{1}{4} A^g_X \cdot \chi - \frac{1}{2\lambda} X \cdot hgrad^g \lambda \cdot \chi - \frac{X(\lambda)}{2\lambda} \chi$$

$$\nabla_U \chi = \nabla^{\mathscr{H}}_U \chi + \frac{1}{4} A^g U \cdot \chi + \frac{1}{4} T^g_U \cdot \chi$$

2. $\dim \mathscr{H}$ ***odd:***

$$\begin{aligned}\nabla_X [\chi, \psi] =& [\nabla^{\mathscr{H}}_X \chi, \nabla^{\mathscr{H}}_X \psi] + \frac{1}{4} A^g_X \cdot [\chi, \psi] \\ & - \frac{1}{2\lambda} [X \cdot hgrad^g \lambda \cdot \chi, X \cdot hgrad^g \lambda \cdot \psi] - \frac{X(\lambda)}{2\lambda} [\chi, \psi]\end{aligned}$$

$$\nabla_U [\chi, \psi] = [\nabla^{\mathscr{H}}_U \chi, \nabla^{\mathscr{H}}_U \psi] + \frac{1}{4} A^g U \cdot [\chi, \psi] + \frac{1}{4} T^g_U \cdot [\chi, \psi]$$

Proof. Calculation using Lemma 5.2.2. □

This has a special case

Theorem 5.3.2. *With the hypotheses and notation of Theorem 5.3.1, now let χ be the basic lift to $\mathcal{M}$ of the spinor field $\check{\chi}$ on the base $\mathcal{B}$ of a conformal submersion with one-dimensional fibres and $\dim \mathcal{B} > 1$. Then for X the basic lift of the vector field $\check{X}$ and U a vertical vector field, the covariant derivatives are given by*

1. $\dim \mathscr{H}$ ***even:***

$$\nabla_X \chi = \check{\nabla}_{\check{X}} \check{\chi} + \frac{1}{4} A^g_X \cdot \chi - \frac{1}{2\lambda} X \cdot hgrad^g \lambda \cdot \chi + \frac{X(\lambda)}{2\lambda} \chi$$

$$\nabla_U \chi = \frac{U(\lambda)}{\lambda} \chi + \frac{1}{4} A^g U \cdot \chi + \frac{1}{4} T^g_U \cdot \chi$$

2. $\dim \mathscr{H}$ ***odd:***

$$\nabla_X [\chi, 0] = [\check{\nabla}_{\check{X}} \check{\chi}, 0] + \frac{1}{4} A^g_X \cdot [\chi, 0] - \frac{1}{2\lambda} [X \cdot hgrad^g \lambda \cdot \chi, 0] + \frac{X(\lambda)}{2\lambda} [\chi, 0]$$

$$\nabla_U [\chi, 0] = \frac{U(\lambda)}{\lambda} [\chi, 0] + \frac{1}{4} A^g U \cdot [\chi, 0] + \frac{1}{4} T^g_U \cdot [\chi, 0]$$

Proof. Apply Proposition 5.2.11 □

As an example of a use of Theorem 5.3.2, consider the case of the Riemannian cone over an even-dimensional manifold. As explained in 3.2 and 3.4, the projection from the cone onto the base is a homothetic submersion with totally geodesic (and therefore totally umbilic) fibres. The tensor field T^g is zero and $A^g_X = -(2/r)X \wedge \partial_r$ where ∂_r is the unit vertical vector field. Theorem 5.3.2 tells us that

$$\nabla_X \chi = \check{\nabla}_{\check{X}} \check{\chi} - \frac{1}{2r} X \wedge \partial_r \cdot \chi$$

and $X \wedge \partial_r \cdot \chi = X \cdot \partial_r \cdot \chi = -\partial_r \cdot X \cdot \chi = -\partial_r \cdot \partial_r \cdot (\check{X} \cdot \check{\chi}) = \check{X} \cdot \check{\chi}$. This reproduces Bär's (real-)Killing spinor relation [Bär93] (although he did not explicitly state it this way) that

$$\nabla_X \chi = 0 \ \forall X \qquad \Longleftrightarrow \qquad \check{\nabla}_{\check{X}} \check{\chi} = \frac{1}{2r} \check{X} \cdot \check{\chi} \ .$$

The odd-dimensional case is similar, but with $[\chi, 0]$ instead of just χ. The other well-known (imaginary-)Killing spinor construction of Baum [Bau89b, Bau89a] may also be reproduced using Theorem 5.3.2. We will not write this as it is similar to Bär's case.

We cannot reproduce the formula relating spinorial covariant derivatives of spinor fields on the base and total space of *generalised cylinders*, as done in [BGM05], because the projection map of such a cylinder is not conformal.

Chapter 6

Dirac Operators and Conformal Submersions

Using the formulae of the previous chapter we can find analogous equations relating the Dirac operator on the total space of a conformal submersion to the Dirac operators of the fibres and base. The calculations are quite lengthy, but will give us some new facts about Dirac morphisms, defined later in the chapter.

6.1 Dirac operator formulae

In this section we will assume $\dim \mathscr{V}, \dim \mathscr{H} > 1$. It is simple to work out the Dirac operator applied to a spinor field on $\mathcal{M}$. First, we recall the following definition and make an observation.

Definition 6.1.1. *The **mean curvature vectors** of $\mathscr{V}$ and $\mathscr{H}$ are*

$$\mu^{\mathscr{V}} \stackrel{\text{def}}{=} \frac{1}{p} tr^{\mathscr{V}} T^g = \frac{1}{p} \sum_{i=1}^{p} T^g_{v_i} v_i \ ,$$
$$\mu^{\mathscr{H}} \stackrel{\text{def}}{=} \frac{1}{n} tr^{\mathscr{H}} A^g = \frac{1}{n} \sum_{i=1}^{n} A^g_{\lambda^{-1} e_i} \lambda^{-1} e_i \ .$$

These definitions are motivated by the role of T as the second fundamental form of the fibres and A playing an analogous role for the base.

Proposition 6.1.2. *The mean curvature vector $\mu^{\mathscr{H}}$ of $\mathscr{H}$ is given by*

$$\mu^{\mathscr{H}} = -\frac{1}{\lambda} vgrad^g \lambda \ .$$

In particular, $\mathscr{H}$ is a minimal distribution if and only if λ is constant on each fibre.

Proof. If V is vertical,

$$\begin{aligned}
0 &= Vg(\lambda^{-1}e_i, \lambda^{-1}e_i) \\
&= V\big[\lambda^{-2}g(e_i, e_i)\big] \\
&= -2\lambda^{-3}V(\lambda)g(e_i, e_i) + \lambda^{-2}Vg(e_i, e_i) \\
&= -2\lambda^{-3}V(\lambda)g(e_i, e_i) - 2\lambda^{-2}g(V, \nabla_{e_i}e_i) \\
&= -2\lambda^{-3}V(\lambda)g(e_i, e_i) - 2\lambda^{-2}g(V, A^g_{e_i}e_i)
\end{aligned}$$

so

$$\frac{V(\lambda)}{\lambda}g(\lambda^{-1}e_i, \lambda^{-1}e_i) + g(V, A^g_{\lambda^{-1}e_i}\lambda^{-1}e_i) = 0\ .$$

The vector $\lambda^{-1}e_i$ has unit norm, so

$$g(V, A^g_{\lambda^{-1}e_i}\lambda^{-1}e_i) = -\frac{V(\lambda)}{\lambda}$$

which is independent of i. Then

$$\mu^{\mathscr{H}} = \frac{1}{n}\sum_{i=1}^{n} A^g_{\lambda^{-1}e_i}\lambda^{-1}e_i = -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda}vgrad^g\lambda = -\frac{1}{\lambda}vgrad^g\lambda\ .$$

$\square$

Theorem 6.1.3. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume* $\dim\mathscr{V}, \dim\mathscr{H} > 1$. *Let φ be a vertical spinor field and let χ and ψ be horizontal spinor fields on $\mathcal{M}$, as in the notation of Theorem 5.1.5, and consider the spinor field on $\mathcal{M}$ constructed from these. The Dirac operators of the total space, base and fibres are related by*

1. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both even****:*

$$\begin{aligned}
\mathcal{D}(\varphi \hat{\otimes} \chi) =& (\hat{\mathcal{D}}\varphi) \hat{\otimes} \chi + \bar{\varphi} \hat{\otimes} (\mathcal{D}^{\mathscr{H}}\chi) + \frac{1}{8}A^g \cdot (\varphi \hat{\otimes} \chi) - \frac{p}{2}\mu^{\mathscr{V}} \cdot (\varphi \hat{\otimes} \chi) \\
&+ \sum_{i=1}^{p}(v_i \cdot \varphi) \hat{\otimes} (\nabla^{\mathscr{H}}_{v_i}\chi) + \sum_{i=1}^{n}(\hat{\nabla}_{\lambda^{-1}e_i}\bar{\varphi}) \hat{\otimes} (\lambda^{-1}e_i) \cdot \chi \\
&+ \frac{n-1}{2\lambda}\bar{\varphi} \hat{\otimes}\ hgrad^g\lambda \cdot \chi\ .
\end{aligned}$$

2. $\dim\mathscr{V}$ ***even*** *and* $\dim\mathscr{H}$ ***odd****: The formula is the same as in the case above, and we must remember that the action of odd elements on χ can be interpreted in two different ways corresponding to the two Clifford representations, and these differ by a minus sign. This means $\mathcal{D}(\varphi \hat{\otimes} \chi)$ has two interpretations.*

3. $\dim\mathscr{V}$ ***odd*** *and* $\dim\mathscr{H}$ ***even****: The formula is the same as in the case above, except with φ and χ swapped as in 3 of Theorem 5.1.5.*

4. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both odd****: In this case* $\varphi^{\perp\xi}$ *is not a vertical spinor field but rather a section of* $\mathbb{S}\mathscr{V}^{\perp\xi}$*, and* ξ *is a unit vertical vector field defined locally as in Theorem 5.1.5.*

$$
\begin{aligned}
\mathcal{D}\big(\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[\chi,\psi]\big) =&(\hat{\mathcal{D}}^{\perp\xi}\varphi^{\perp\xi})\mathbin{\hat{\otimes}}[\psi,\chi]+\overline{\varphi^{\perp\xi}}\mathbin{\hat{\otimes}}[\mathcal{D}^{\mathscr{H}}\psi,\mathcal{D}^{\mathscr{H}}\chi]\\
&+\sum_{i=2}^{p}(v_i\cdot\varphi^{\perp\xi})\mathbin{\hat{\otimes}}[\nabla^{\mathscr{H}}_{v_i}\psi,\nabla^{\mathscr{H}}_{v_i}\chi]\\
&+\sum_{i=1}^{n}(\hat{\nabla}^{\perp\xi}_{\lambda^{-1}e_i}\overline{\varphi^{\perp\xi}})\mathbin{\hat{\otimes}}[\lambda^{-1}e_i\cdot\psi,\lambda^{-1}e_i\cdot\chi]\\
&+\frac{1}{8}A^g\cdot\big(\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[\chi,\psi]\big)-\frac{p}{2}\mu^{\mathscr{V}}\cdot\big(\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[\chi,\psi]\big)\\
&+\frac{n-1}{2\lambda}\overline{\varphi^{\perp\xi}}\mathbin{\hat{\otimes}}[hgrad^g\lambda\cdot\psi,hgrad^g\lambda\cdot\chi]\\
&-i^{\dim\mathcal{M}}\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[-\nabla^{\mathscr{H}}_{\xi}\psi,\nabla^{\mathscr{H}}_{\xi}\chi]-i^{\dim\mathcal{M}}(\hat{\nabla}^{\perp\xi}_{\xi}\varphi^{\perp\xi})\mathbin{\hat{\otimes}}[-\psi,\chi]\\
&-\frac{1}{2}i^{\dim\mathcal{M}}\hat{\nabla}\xi\cdot\big(\varphi^{\perp\xi}\mathbin{\hat{\otimes}}[\psi,-\chi]\big)\ .
\end{aligned}
$$

Proof. For *1*, we'll look at the parts involving the vertical frame v and the parts involving the horizontal frame $\lambda^{-1}e$ separately to keep things simpler.

$$
\begin{aligned}
\mathcal{D}^v(\varphi\mathbin{\hat{\otimes}}\chi)=&\sum_{i=1}^{p}v_i\cdot(\hat{\nabla}_{v_i}\varphi\mathbin{\hat{\otimes}}\chi)+\sum_{i=1}^{p}v_i\cdot(\varphi\mathbin{\hat{\otimes}}\nabla^{\mathscr{H}}_{v_i}\chi)\\
&+\frac{1}{4}\sum_{i=1}^{p}v_i\cdot(\varphi\mathbin{\hat{\otimes}}A^g v_i\chi)+\frac{1}{4}\sum_{i=1}^{p}v_i\cdot T^g_{v_i}\cdot(\varphi\mathbin{\hat{\otimes}}\chi)\ .
\end{aligned}
$$

The first term is obviously $(\hat{\mathcal{D}}\varphi)\mathbin{\hat{\otimes}}\chi$. The third term requires a calculation. Since A^g swaps $\mathscr{V}$ and $\mathscr{H}$, we can split A^g into a part A_1 sending $\mathscr{V}$ to $\mathscr{H}$ and a part A_2 going the other way. So $A^g=A_1+A_2$ where

$$
\begin{aligned}
A_1&=\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{p}A_1^{ijk}(\lambda^{-1}e_i)\otimes(\lambda^{-1}e_j)\otimes v_k\ ,\\
A_2&=\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{p}A_2^{ijk}(\lambda^{-1}e_i)\otimes v_k\otimes(\lambda^{-1}e_j)\ ,
\end{aligned}
$$

where A_1^{ijk} and A_2^{ijk} are both skew in i,j. Then

$$
g(A_XY,U)=g(\nabla_XY,U)=-g(Y,\nabla_XU)=-g(Y,A_XU)
$$

implies that $A_2^{ijk}=-A_1^{ijk}$. If we write $A_1^{ijk}=A^{ijk}$, then

$$
\begin{aligned}
A^g&=\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{p}A^{ijk}(\lambda^{-1}e_i)\otimes\big[(\lambda^{-1}e_j)\otimes v_k-v_k\otimes(\lambda^{-1}e_j)\big]\\
&=\sum_{i,j=1}^{n}\sum_{k=1}^{p}A^{ijk}(\lambda^{-1}e_i)\otimes(\lambda^{-1}e_j)\wedge v_k
\end{aligned}
$$

and this is skew in i, j. Now

$$\begin{aligned}A^g v_k \cdot \chi &= -\frac{1}{2} \sum_{i,j=1}^{n} A^{ijk} (\lambda^{-1} e_i) \otimes (\lambda^{-1} e_j) \cdot \chi \\ &= -\frac{1}{2} \sum_{i,j=1}^{n} A^{ijk} (\lambda^{-1} e_i)(\lambda^{-1} e_j) \cdot \chi \ ,\end{aligned}$$

so

$$\begin{aligned}\sum_{k=1}^{p} v_k \cdot (\varphi \mathbin{\hat{\otimes}} A^g v_k \chi) &= \sum_{k=1}^{p} (v_k \cdot \varphi) \mathbin{\hat{\otimes}} A^g v_k \chi \\ &= -\frac{1}{2} \sum_{k=1}^{p} (v_k \cdot \varphi) \mathbin{\hat{\otimes}} \sum_{i,j=1}^{n} A^{ijk} (\lambda^{-1} e_i)(\lambda^{-1} e_j) \cdot \chi \ .\end{aligned}$$

Also

$$\begin{aligned}A^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi) &= \sum_{i,j=1}^{n} \sum_{k=1}^{p} A^{ijk} (\lambda^{-1} e_i) \otimes (\lambda^{-1} e_j) \wedge v_k \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ &= \sum_{i,j=1}^{n} \sum_{k=1}^{p} A^{ijk} (\lambda^{-1} e_i)(\lambda^{-1} e_j) v_k \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ &= \sum_{i,j=1}^{n} \sum_{k=1}^{p} A^{ijk} (v_k \cdot \varphi) \mathbin{\hat{\otimes}} (\lambda^{-1} e_i)(\lambda^{-1} e_j) \cdot \chi \ .\end{aligned}$$

Thus $\sum_{k=1}^{p} v_k \cdot (\varphi \mathbin{\hat{\otimes}} A^g v_k \chi) = -\frac{1}{2} A^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi)$. A similar procedure allows us to write

$$T^g = \sum_{i,j=1}^{p} \sum_{k=1}^{n} T^{ijk} v_i \otimes v_j \wedge (\lambda^{-1} e_k)$$

and this is symmetric in i, j. Then

$$\begin{aligned}\sum_{i=1}^{p} v_i \cdot T^g_{v_i} \cdot (\varphi \mathbin{\hat{\otimes}} \chi) &= \sum_{i=1}^{p} v_i \cdot \sum_{j=1}^{p} \sum_{k=1}^{n} T^{ijk} v_j (\lambda^{-1} e_k) \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \\ &= \sum_{i,j=1}^{p} \sum_{k=1}^{n} T^{ijk} (v_i v_j \cdot \bar{\varphi}) \mathbin{\hat{\otimes}} (\lambda^{-1} e_k) \cdot \chi \ .\end{aligned}$$

But $T^{ijk} = T^{jik}$ and $v_i v_j = -v_j v_i$ for $i \neq j$. So all that remains is

$$\sum_{i=1}^{p} \sum_{k=1}^{n} T^{iik} (v_i v_i \cdot \bar{\varphi}) \mathbin{\hat{\otimes}} (\lambda^{-1} e_k) \cdot \chi = -\sum_{i=1}^{p} \sum_{k=1}^{n} T^{iik} \bar{\varphi} \mathbin{\hat{\otimes}} (\lambda^{-1} e_k) \cdot \chi$$

and this is exactly $-2p\mu^{\mathscr{V}} \cdot (\varphi \mathbin{\hat{\otimes}} \chi)$. So far we have

$$\begin{aligned}\mathcal{D}^v (\varphi \mathbin{\hat{\otimes}} \chi) =& (\hat{\mathcal{D}} \varphi) \mathbin{\hat{\otimes}} \chi + \sum_{i=1}^{p} v_i \cdot (\varphi \mathbin{\hat{\otimes}} \nabla^{\mathscr{H}}_{v_i} \chi) \\ &- \frac{1}{8} A^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi) - \frac{p}{2} \mu^{\mathscr{V}} \cdot (\varphi \mathbin{\hat{\otimes}} \chi) \ .\end{aligned}$$

Now we shall look at the horizontal parts. We have

$$\begin{aligned}\mathcal{D}^h(\varphi \,\hat{\otimes}\, \chi) = &\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot(\hat{\nabla}_{\lambda^{-1}e_i}\varphi \,\hat{\otimes}\, \chi) + \sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot(\varphi \,\hat{\otimes}\, \nabla^{\mathscr{H}}_{\lambda^{-1}e_i}\chi)\\ &+ \frac{1}{4}\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot A^g_{\lambda^{-1}e_i}\cdot(\varphi \,\hat{\otimes}\, \chi)\\ &- \frac{1}{2\lambda}\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot(\varphi \,\hat{\otimes}\, \lambda^{-1}e_i\cdot hgrad^g\lambda\cdot\chi)\\ &- \sum_{i=1}^{n}\frac{\lambda^{-1}e_i(\lambda)}{2\lambda}(\lambda^{-1}e_i)\cdot(\varphi \,\hat{\otimes}\, \chi)\ .\end{aligned}$$

The second term is clearly $\bar{\varphi} \,\hat{\otimes}\, (\mathcal{D}^{\mathscr{H}}\chi)$. The calculation above also gives us

$$\begin{aligned}\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot A^g_{\lambda^{-1}e_i}\cdot(\varphi \,\hat{\otimes}\, \chi) &= \sum_{i,j=1}^{n}\sum_{k=1}^{p}A^{ijk}(\lambda^{-1}e_i)(\lambda^{-1}e_j)v_k\cdot(\varphi \,\hat{\otimes}\, \chi)\\ &= A^g\cdot(\varphi \,\hat{\otimes}\, \chi)\ .\end{aligned}$$

The other parts are easy:

$$\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot(\varphi \,\hat{\otimes}\, \lambda^{-1}e_i\cdot hgrad^g\lambda\cdot\chi) = -\sum_{i=1}^{n}\bar{\varphi} \,\hat{\otimes}\, hgrad^g\lambda\cdot\chi = -n\bar{\varphi} \,\hat{\otimes}\, hgrad^g\lambda\cdot\chi\ ,$$

and

$$\sum_{i=1}^{n}\frac{\lambda^{-1}e_i(\lambda)}{2\lambda}(\lambda^{-1}e_i)\cdot(\varphi \,\hat{\otimes}\, \chi) = \frac{1}{2\lambda}\bar{\varphi} \,\hat{\otimes}\, hgrad^g\lambda\cdot\chi\ .$$

We get

$$\begin{aligned}\mathcal{D}^h(\varphi \,\hat{\otimes}\, \chi) = &\sum_{i=1}^{n}(\lambda^{-1}e_i)\cdot(\hat{\nabla}_{\lambda^{-1}e_i}\varphi \,\hat{\otimes}\, \chi) + \bar{\varphi} \,\hat{\otimes}\, (\mathcal{D}^{\mathscr{H}}\chi)\\ &+ \frac{1}{4}A^g\cdot(\varphi \,\hat{\otimes}\, \chi) + \frac{n-1}{2\lambda}\bar{\varphi} \,\hat{\otimes}\, hgrad^g\lambda\cdot\chi\ .\end{aligned}$$

Adding up all the terms gives part *1* of the theorem. For part *4*, the proof is similar but

$$\hat{\mathcal{D}}^{\perp\xi}\varphi^{\perp\xi} = \sum_{i=2}^{p}v_i\cdot\hat{\nabla}^{\perp\xi}_{v_i}\varphi^{\perp\xi}$$

so we get an extra term corresponding to $v_1 = \xi$, which is

$$-i^{\dim\mathcal{M}}(\hat{\nabla}^{\perp\xi}_{\xi}\varphi^{\perp\xi}) \,\hat{\otimes}\, \big[-\psi,\chi\big]\ .$$

Also

$$\begin{aligned}\sum_{i=1}^{p}v_i\cdot\big(\varphi^{\perp\xi} \,\hat{\otimes}\, \big[\nabla^{\mathscr{H}}_{v_i}\chi,\nabla^{\mathscr{H}}_{v_i}\psi\big]\big) = &\sum_{i=2}^{p}(v_i\cdot\varphi^{\perp\xi}) \,\hat{\otimes}\, \big[\nabla^{\mathscr{H}}_{v_i}\psi,\nabla^{\mathscr{H}}_{v_i}\chi\big]\\ &- i^{\dim\mathcal{M}}\varphi^{\perp\xi} \,\hat{\otimes}\, \big[-\nabla^{\mathscr{H}}_{\xi}\psi,\nabla^{\mathscr{H}}_{\xi}\chi\big]\ .\end{aligned}$$

Finally, the extra terms in the covariant derivative formulae give

$$\begin{aligned}
&-\frac{1}{2}i^{\dim \mathcal{M}}\sum_{i=1}^{p} v_i \cdot \hat{\nabla}_{v_i}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[\psi, -\chi\right]\right) \\
&-\frac{1}{2}i^{\dim \mathcal{M}}\sum_{i=1}^{n} (\lambda^{-1}e_i) \cdot \hat{\nabla}_{\lambda^{-1}e_i}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[\psi, -\chi\right]\right) \\
&= -\frac{1}{2}i^{\dim \mathcal{M}}\hat{\nabla}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[\psi, -\chi\right]\right) .
\end{aligned}$$

This concludes the proof. □

Lemma 6.1.4. *With respect to the natural isomorphism of vector bundles associated to $Spin(\mathscr{V}^{\perp\xi})$*

$$\mathbb{S}\mathscr{V}^{\perp\xi} \cong \mathbb{S}\mathscr{V}$$

we can write

$$(\xi \star \hat{\mathcal{D}}\varphi)^{\perp\xi} = -\hat{\mathcal{D}}^{\perp\xi}\varphi^{\perp\xi} - (\hat{\nabla}_{\xi}\varphi)^{\perp\xi} + \frac{1}{2}\sum_{i=2}^{p} v_i \cdot \hat{\nabla}_{v_i}\xi \cdot \varphi^{\perp\xi}$$

where $\varphi^{\perp\xi}$ is a section of $\mathbb{S}\mathscr{V}^{\perp\xi} \cong \mathbb{S}\mathscr{V}$ and $\xi\star$ is the illegal action of ξ.

Proof. Simple calculation using Lemma 5.2.6. □

Lemma 6.1.5. *In the formula for $\mathcal{D}\left(\varphi^{\perp\xi} \hat{\otimes} \left[\chi, \psi\right]\right)$ in part 4 of Theorem 6.1.3, the terms*

$$\begin{aligned}
&(\hat{\mathcal{D}}^{\perp\xi}\varphi^{\perp\xi}) \hat{\otimes} \left[\psi, \chi\right] + \sum_{i=1}^{n}(\hat{\nabla}^{\perp\xi}_{\lambda^{-1}e_i}\overline{\varphi^{\perp\xi}}) \hat{\otimes} \left[\lambda^{-1}e_i \cdot \psi, \lambda^{-1}e_i \cdot \chi\right] \\
&\quad - i^{\dim \mathcal{M}}(\hat{\nabla}^{\perp\xi}_{\xi}\varphi^{\perp\xi}) \hat{\otimes} \left[-\psi, \chi\right] - \frac{1}{2}i^{\dim \mathcal{M}}\hat{\nabla}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[\psi, -\chi\right]\right)
\end{aligned}$$

are equal to, for $\dim \mathcal{M} = 0 \bmod 4$

$$\begin{aligned}
&-(\xi \star \hat{\mathcal{D}}\varphi)^{\perp\xi} \hat{\otimes} \left[\psi, \chi\right] + \sum_{i=1}^{n}\overline{(\hat{\nabla}_{\lambda^{-1}e_i}\varphi)^{\perp\xi}} \hat{\otimes} \left[\lambda^{-1}e_i \cdot \psi, \lambda^{-1}e_i \cdot \chi\right] \\
&\quad - 2(\hat{\nabla}_{\xi}\varphi)^{\perp\xi} \hat{\otimes} \left[0, \chi\right] + \hat{\nabla}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[0, \chi\right]\right) .
\end{aligned}$$

and for $\dim \mathcal{M} = 2 \bmod 4$

$$\begin{aligned}
&-(\xi \star \hat{\mathcal{D}}\varphi)^{\perp\xi} \hat{\otimes} \left[\psi, \chi\right] + \sum_{i=1}^{n}\overline{(\hat{\nabla}_{\lambda^{-1}e_i}\varphi)^{\perp\xi}} \hat{\otimes} \left[\lambda^{-1}e_i \cdot \psi, \lambda^{-1}e_i \cdot \chi\right] \\
&\quad - 2(\hat{\nabla}_{\xi}\varphi)^{\perp\xi} \hat{\otimes} \left[\psi, 0\right] + \hat{\nabla}\xi \cdot \left(\varphi^{\perp\xi} \hat{\otimes} \left[\psi, 0\right]\right) .
\end{aligned}$$

Proof. This is a calculation, using both Lemma 5.2.6 and Lemma 6.1.4 and reassembling some terms to make $\hat{\nabla}\xi$. □

Just as we did for covariant derivatives in Section 5.2, we want to know under what conditions some of the terms in the above expressions vanish. If $\dim \mathcal{M} = 0 \bmod 4$ it is clear that putting the positive chiral part $\chi = 0$ will get rid of some terms, and if $\dim \mathcal{M} = 2 \bmod 4$ putting the positive chiral part $\psi = 0$ will do the trick. Exactly as in the case for covariant derivatives, *when we take the basic lift of a spinor field when* $\dim \mathscr{V}$ *and* $\dim \mathscr{H}$ *are both odd, we always define the lift to be of negative chirality.*

Lemma 6.1.6. *When χ is the basic lift of $\check{\chi}$,*

$$\mathcal{D}^{\mathscr{H}}\chi = \lambda^{-1}\check{\mathcal{D}}\check{\chi} + \lambda^{-1}hgrad^{g}\lambda \cdot \chi \ .$$

In this equation $\check{\mathcal{D}}\check{\chi}$ refers to the basic lift to $\mathcal{M}$ but we have omitted this notation.

Proof. Use 5.2.11. □

We can now state the relations between the Dirac operators when χ is a basic spinor field. We shall do this in all four cases.

Corollary 6.1.7. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim\mathscr{V}, \dim\mathscr{H} > 1$. Let φ be a vertical spinor field and let χ be the basic lift of the spinor field $\check{\chi}$, and consider the spinor field on $\mathcal{M}$ constructed from these. The Dirac operators of the total space, base and fibres are related by*

1. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both even****:*

$$\begin{aligned}\mathcal{D}(\varphi \mathbin{\hat{\otimes}} \chi) =&(\hat{\mathcal{D}}\varphi) \mathbin{\hat{\otimes}} \chi + \frac{1}{\lambda}\bar{\varphi} \mathbin{\hat{\otimes}} (\check{\mathcal{D}}\check{\chi}) + \frac{1}{8}A^g \cdot (\varphi \mathbin{\hat{\otimes}} \chi)\\ &- \frac{p}{2}\mu^{\mathscr{V}} \cdot (\varphi \mathbin{\hat{\otimes}} \chi) - \mu^{\mathscr{H}} \cdot (\varphi \mathbin{\hat{\otimes}} \chi)\\ &+ \sum_{i=1}^{n}(\hat{\nabla}_{\lambda^{-1}e_i}\bar{\varphi}) \mathbin{\hat{\otimes}} (\lambda^{-1}e_i) \cdot \chi + \frac{n+1}{2\lambda}\bar{\varphi} \mathbin{\hat{\otimes}} hgrad^{g}\lambda \cdot \chi \ .\end{aligned}$$

2. $\dim\mathscr{V}$ ***even*** *and* $\dim\mathscr{H}$ ***odd****: The formula is the same as in the case above, and we must remember that the action of odd elements on χ can be interpreted in two different ways corresponding to the two Clifford representations, and these differ by a minus sign. This means $\mathcal{D}(\varphi \mathbin{\hat{\otimes}} \chi)$ has two interpretations.*

3. $\dim\mathscr{V}$ ***odd*** *and* $\dim\mathscr{H}$ ***even****: The formula is the same as in the case above, except with φ and χ swapped as in 3 of Theorem 5.1.5.*

4. $\dim\mathscr{V}$ *and* $\dim\mathscr{H}$ ***both odd****,* $\dim\mathcal{M} = 2 \bmod 4$*: Now φ is a vertical spinor field and χ is horizontal, and Clifford multiplication has been extended from $\mathbb{S}\mathscr{V}^{\perp\xi}$ by $\cdot_{p2}$ to produce $\mathbb{S}\mathscr{V}$.*

$$\begin{aligned}\mathcal{D}\big(\varphi^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, 0]\big) =& -(\xi \star \hat{\mathcal{D}}\varphi)^{\perp\xi} \mathbin{\hat{\otimes}} [0, \chi] + \frac{1}{\lambda}\overline{\varphi^{\perp\xi}} \mathbin{\hat{\otimes}} [0, \check{\mathcal{D}}\check{\chi}]\\ &+ \frac{1}{8}A^g \cdot \big(\varphi^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, 0]\big)\\ &- \frac{p}{2}\mu^{\mathscr{V}} \cdot \big(\varphi^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, 0]\big) - \mu^{\mathscr{H}} \cdot \big(\varphi^{\perp\xi} \mathbin{\hat{\otimes}} [\chi, 0]\big)\\ &+ \sum_{i=1}^{n}\overline{(\hat{\nabla}_{\lambda^{-1}e_i}\varphi)^{\perp\xi}} \mathbin{\hat{\otimes}} [0, \lambda^{-1}e_i \cdot \chi]\\ &+ \frac{n+1}{2\lambda}\overline{\varphi^{\perp\xi}} \mathbin{\hat{\otimes}} [0, hgrad^{g}\lambda \cdot \chi] \ .\end{aligned}$$

When $\dim\mathcal{M} = 0 \bmod 4$*, the equation is the same except we use $[0, \psi]$ throughout instead.*

Proof. This follows easily from Proposition 5.2.11, Proposition 6.1.2 and Lemma 6.1.6. The factor $n+1$ having replaced a factor $n-1$ previously is not an error, and follows from Lemma 6.1.6. □

6.2 One-dimensional fibres

For completeness we now include the Dirac operator formulae in the case $\dim \mathscr{V} = 1$ and $\dim \mathscr{H} > 1$.

Theorem 6.2.1. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with conformal factor λ and O'Neill's tensor fields T^g and A^g on $\mathcal{M}$. Denote by $\mathscr{V}$ and $\mathscr{H}$ the vertical and horizontal distributions and assume $\dim \mathscr{V} = 1$ and $\dim \mathscr{H} > 1$. Let χ and ψ be horizontal spinor fields on $\mathcal{M}$. The Dirac operators are related by*

1. $\dim \mathscr{H}$ ***even:***

$$\mathcal{D}\chi = \mathcal{D}^{\mathscr{H}}\chi + \frac{1}{8}A^g \cdot \chi - \frac{1}{2}\mu^{\mathscr{V}} \cdot \chi + v_1 \cdot \nabla^{\mathscr{H}}_{v_1}\chi + \frac{n+1}{2\lambda} hgrad^g\lambda \cdot \chi$$

2. $\dim \mathscr{H}$ ***odd:***

$$\begin{aligned}\mathcal{D}\big[\chi, \psi\big] =& \big[\mathcal{D}^{\mathscr{H}}\psi, \mathcal{D}^{\mathscr{H}}\chi\big] + \frac{1}{8}A^g \cdot \big[\chi, \psi\big] - \frac{1}{2}\mu^{\mathscr{V}} \cdot \big[\chi, \psi\big] \\ &+ v_1 \cdot \big[\nabla^{\mathscr{H}}_{v_1}\chi, \nabla^{\mathscr{H}}_{v_1}\psi\big] + \frac{n+1}{2\lambda}\big[hgrad^g\lambda \cdot \psi, hgrad^g\lambda \cdot \chi\big]\end{aligned}$$

Proof. These follow from Theorem 5.3.1. □

Theorem 6.2.2. *Let χ be the basic lift to $\mathcal{M}$ of the spinor field $\check{\chi}$ on the base $\mathcal{B}$ of a conformal submersion with one-dimensional fibres and $\dim \mathcal{B} > 1$. Then the Dirac operators are related by*

1. $\dim \mathscr{H}$ ***even:***

$$\mathcal{D}\chi = \frac{1}{\lambda}\check{\mathcal{D}}\check{\chi} + \frac{1}{8}A^g \cdot \chi - \frac{1}{2}\mu^{\mathscr{V}} \cdot \chi - \mu^{\mathscr{H}} \cdot \chi + \frac{n+1}{2\lambda} hgrad^g\lambda \cdot \chi$$

2. $\dim \mathscr{H}$ ***odd:***

$$\begin{aligned}\mathcal{D}\big[\chi, 0\big] =& \big[0, \mathcal{D}^{\mathscr{H}}\chi\big] + \frac{1}{8}A^g \cdot \big[\chi, 0\big] - \frac{1}{2}\mu^{\mathscr{V}} \cdot \big[\chi, 0\big] \\ &- \mu^{\mathscr{H}} \cdot \big[\chi, 0\big] + \frac{n+1}{2\lambda}\big[0, hgrad^g\lambda \cdot \chi\big]\end{aligned}$$

Proof. These follow from Theorem 5.3.2. □

It is interesting to note that

Proposition 6.2.3. *In the case of a conformal submersion with one-dimensional fibres, we can consider the fibres to be smooth curves parametrised by arc length. The geodesic curvature of these curves is equal to the mean curvature vector of the fibres considered as isometrically embedded submanifolds.*

Proof. Since we are parametrising the fibres by arc length, their tangent vectors are given by v_1, up to sign.

$$\mu^{\mathscr{V}} = T_{v_1}v_1 = h\nabla_{v_1}v_1 \ .$$

Also because we are parametrising the fibres by arc length, $\hat{\nabla}v_1 = 0$, and

$$v\nabla_{v_1}v_1 = v\nabla_{v_1}vv_1 = \hat{\nabla}_{v_1}v_1 = 0$$

so

$$\mu^{\mathscr{V}} = h\nabla_{v_1} v_1 = \nabla_{v_1} v_1$$

which is the geodesic curvature. □

This allows us to simplify Theorem 6.2.2 when the fibres are geodesics. Note that an embedded curve is a totally geodesic submanifold if and only if it is a geodesic, so the analogous property in high fibre dimensions is $T^g = 0$.

6.3 Dirac morphisms

The theory of harmonic maps between Riemannian manifolds has been studied for some time since the foundational paper of Eells and Sampson [ES64]. A harmonic map between Riemannian manifolds is a smooth map which is an extremum of the Dirichlet energy functional. Equivalently, it lies in the kernel of the induced Laplacian on the space of smooth maps between Riemannian manifolds.

Theorem 6.3.1. *([Wat73]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a smooth surjective map between Riemannian manifolds. Then π commutes with the Laplacian (acting on functions) if and only if π is a harmonic Riemannian submersion.*

Remark 6.3.2. *We do not need to assume π is a submersion; Watson proves that π is Riemannian, which implies that $\mathscr{H} \to T\mathcal{B}$ is pointwise-injective and therefore surjective, so π is a submersion. Watson [Wat73] assumes in his paper that $\mathcal{M}$ is compact, but comments that the above holds when $\mathcal{M}$ and $\mathcal{B}$ are merely complete. However, completeness is not used in the proof of the theorem. In the same paper, analogous results are shown for maps commuting with the Laplacian acting on higher-degree forms.*

An interesting fact is

Theorem 6.3.3. *([Wat73]) A Riemannian submersion is harmonic if and only if its fibres are totally geodesic, i.e. $T = 0$.*

It is clear that a map that commutes with the Laplacian preserves all of its eigenspaces. The condition that only the kernel is preserved is weaker.

Definition 6.3.4. *([Fug78]) A* ***harmonic morphism*** *is a smooth map $\mathcal{M} \to \mathcal{B}$ such that the pullback of a harmonic function on $\mathcal{B}$ is harmonic on $\mathcal{M}$.*

Note that if $\dim \mathcal{B} > \dim \mathcal{M}$ then any harmonic morphisms are constant. Harmonic morphisms have been characterised by Ishihara, who proved

Theorem 6.3.5. *([Ish79])*[1] *A smooth surjective map between Riemannian manifolds is a harmonic morphism if and only if it is a harmonic conformal pseudo-submersion.*

Remark 6.3.6. *A* pseudo-submersion *as defined by Ishihara is a smooth surjective map whose derivative is pointwise either surjective or zero. Fuglede calls these* semiconformal mappings. *We shall ignore this technicality as it will not be relevant for us.*

[1] This theorem was proved independently by Fuglede [Fug78].

Thus harmonic morphisms can be thought of as conformal analogues of maps that commute with the Laplacian.
The paper [LS09] by Loubeau and Slobodeanu attempts to characterise those maps which preserve the germ[2] of the *Dirac operator*. They make a definition equivalent to the following one.

Definition 6.3.7. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal[3] submersion of Riemannian spin manifolds with fibres of dimension greater than* 1*. Then π is a **Dirac morphism** if there exists a vertical spinor field φ on $\mathcal{M}$ satisfying*

1. *φ is parallel in horizontal directions: $\hat{\nabla}_X \varphi = 0$ for all horizontal X,*
2. *$\hat{\mathcal{D}}\varphi = \left(1 + \frac{n}{4}\right)\mu^{\mathscr{H}} \cdot \varphi$*

and, if $\check{\chi}$ is a harmonic spinor field on the base $\mathcal{B}$ then $\varphi \mathbin{\hat{\otimes}} \chi$ is harmonic on $\mathcal{M}$, where χ is the basic lift of $\check{\chi}$.

There are several remarks to make about this definition. Firstly, it is not made clear in [LS09] how strict the conditions on the vertical spinor φ are. Secondly, the authors restrict their attention to the case where $\mathcal{B}$ is even-dimensional. It is not clear why this restriction is made, although it is probably because the spinor bundles are easier to understand in this case as we have seen. On the positive side, if we can find two vertical spinors satisfying the above two conditions then π will be a Dirac morphism with respect to one if and only if it is also a Dirac morphism with respect to the other. This means the notion of Dirac morphism does actually make sense. We should also say that the notation of the paper [LS09] differs considerably from that of this thesis, and some of the conventions used will make certain expressions appear to disagree, although they can be seen to agree qualitatively. Loubeau and Slobodeanu offer the following main result.

Theorem 6.3.8. *([LS09]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with fibres of dimension greater than* 1 *and where* $\dim \mathcal{B}$ *is **even**, and let φ be a vertical spinor field satisfying the conditions in Definition 6.3.7. Then π is a Dirac morphism if and only if its horizontal distribution is integrable and*

$$p\mu^{\mathscr{V}} - \frac{n+1}{\lambda} hgrad^g \lambda = 0 \ .$$

The proof of this theorem uses a formula for the Dirac operator similar to our own, and demands that

$$\mathcal{D}(\varphi \mathbin{\hat{\otimes}} \chi) = \lambda^{-1} \bar{\varphi} \mathbin{\hat{\otimes}} (\check{\mathcal{D}}\check{\chi}) \ .$$

The paper [LS09] does not derive formulae for the covariant derivatives directly, but skips straight to relating the Dirac operators involved. We can give a similar result for the case when $\mathcal{B}$ has odd dimension, and in fact we have also done enough to reproduce the theorem above. This comes from our formulae for the Dirac operators in Corollary 6.1.7. Since we have already stated the known result for $\dim \mathcal{B}$ even, here is the new odd version.

[2] Due to the comparison between the notions of harmonic mapping and harmonic morphism, such a map would be better termed a *Dirac mapping* and the term *Dirac morphism* reserved for those maps which preserve only the kernel rather than the entire spectrum.

[3] It is necessary include 'conformal' in the hypotheses here because otherwise there is no natural way to compare the spinor bundles.

Theorem 6.3.9. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with fibres of dimension greater than* 1 *and where* $\dim \mathcal{B}$ *is* ***odd****, and let φ be a vertical spinor field satisfying the conditions in Definition 6.3.7. Then π is a Dirac morphism if and only if its horizontal distribution is integrable and*

$$p\mu^{\mathscr{V}} - \frac{n+1}{\lambda} hgrad^g \lambda = 0 \ .$$

Proof. Recall that the horizontal distribution $\mathscr{H}$ is integrable if and only if $A^{\lambda^{-2}g} = 0$ where $\lambda^{-2}g$ is the metric on $\mathcal{M}$ conformal to g which makes π Riemannian. Since

$$A_X^{\lambda^{-2}g} = A_X^g + \frac{2}{\lambda} X \wedge vgrad^g \lambda$$

we have

$$\begin{aligned} A^{\lambda^{-2}g} &= \sum_{i=1}^{n} (\lambda^{-1}e_i) \otimes A^{\lambda^{-2}g}_{\lambda^{-1}e_i} \\ &= A^g + \frac{2}{\lambda} \sum_{i=1}^{n} (\lambda^{-1}e_i) \otimes (\lambda^{-1}e_i) \wedge vgrad^g \lambda \end{aligned}$$

and so the Clifford actions satisfy

$$\begin{aligned} A^g \cdot &= A^{\lambda^{-2}g} \cdot - \frac{2}{\lambda} \sum_{i=1}^{n} (\lambda^{-1}e_i)^2 vgrad^g \lambda \cdot \\ &= A^{\lambda^{-2}g} \cdot + \frac{2n}{\lambda} vgrad^g \lambda \cdot \\ &= A^{\lambda^{-2}g} \cdot - 2n\mu^{\mathscr{H}} \cdot \ . \end{aligned}$$

It is clear from Corollary 6.1.7 that π is a Dirac morphism if and only if

$$\begin{aligned} \hat{\mathcal{D}}\varphi &= \big(1 + \frac{n}{4}\big)\mu^{\mathscr{H}} \cdot \varphi \ , \\ \hat{\nabla}_{\lambda^{-1}e_i}\varphi &= 0 \qquad \forall i \end{aligned}$$

and the Clifford action of

$$\frac{1}{8}A^{\lambda^{-2}g} - \frac{p}{2}\mu^{\mathscr{V}} + \frac{n+1}{2\lambda} hgrad^g \lambda$$

is zero, i.e. $A^{\lambda^{-2}g} = 0$ and

$$p\mu^{\mathscr{V}} - \frac{n+1}{\lambda} hgrad^g \lambda = 0 \ .$$

□

The definition of Dirac morphism is simpler when the fibres are one-dimensional.

Definition 6.3.10. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with fibres of dimension* 1*. Then π is a* ***Dirac morphism*** *if whenever $\check{\chi}$ is a harmonic spinor field on the base $\mathcal{B}$ then its basic lift χ is harmonic on $\mathcal{M}$.*

In the paper [LS09] a result is also given for this simpler situation:

Theorem 6.3.11. *([LS09]) Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with fibres of dimension* 1 *and where* $\dim \mathcal{B}$ *is* ***even****. Then π is a Dirac morphism if* *and only if its horizontal distribution is integrable* and *minimal, and*

$$\mu^{\mathscr{V}} - \frac{n+1}{\lambda} hgrad^g \lambda = 0 \ .$$

We can prove the other half of the theorem.

Theorem 6.3.12. *Let $\pi : \mathcal{M} \to \mathcal{B}$ be a conformal submersion of Riemannian spin manifolds with fibres of dimension* 1 *and where* $\dim \mathcal{B}$ *is* ***odd****. Then π is a Dirac morphism if and only if its horizontal distribution is integrable* and *minimal, and*

$$\mu^{\mathscr{V}} - \frac{n+1}{\lambda} hgrad^g \lambda = 0 \ .$$

Proof. The proof is a simpler version of that for Theorem 6.3.9 and uses Theorem 6.2.2. □

It is easy to see from the above theorems what the corresponding statements are when π is a Riemannian submersion. We will not state these here although they do appear, in the case $\dim \mathcal{B}$ is even, in [LS09]. For one-dimensional fibres, this characterisation of Dirac morphisms is not given when $\dim \mathcal{B}$ is odd in [LS09] even though the relevant algebraic construction appears for example in the paper of Bär, Gauduchon and Moroianu [BGM05]. These authors did not consider Dirac operators or the case of higher fibre dimension, however.

Remark 6.3.13. *The covariant derivative on the spinor bundle of a manifold $\mathcal{M}^n$ splits into irreducibles (with respect to the spin group)*

$$\nabla = -\frac{1}{n} g \cdot \mathcal{D} + \mathcal{T}$$

where $\mathcal{D}$ is the Dirac operator and $\mathcal{T}$ is the twistor operator. We will see this decomposition again in 7.4 and 8.3. It is possible to write down formulae relating the twistor operators of the total space, base and fibres of a conformal submersion of Riemannian spin manifolds. One can then define a twistor morphism *to be such a map which preserves the germ of the twistor operator, in the same way Dirac morphisms are defined. However, we do not include these relations because such maps do not exist in non-trivial cases.*

Chapter 7

G_2-structures

One of the two exceptional restricted holonomy groups of locally irreducible non-locally symmetric spaces appearing on Berger's list is the Lie group G_2. Manifolds with G_2 holonomy were not known to exist until the celebrated work of Bryant [Bry87], in which local existence was established. The later paper [BS89], coauthored with Salamon, showed that such metrics existed on complete spaces and, later still, Joyce [Joy96b] proved that there are compact manifolds with holonomy G_2. A natural generalisation of these spaces are those possessing a G_2-structure. These were studied and divided into sixteen classes by Fernández and Gray [FG82], and it is known that exactly fifteen of these exist. In this chapter, we reproduce this classification and show that it coincides with another classification in terms of naturally occurring spinor fields. We expect these two classifications to coincide due to Theorem D.0.39, and we show this explicitly by deriving relationships between the covariant derivatives of the relevant canonical tensor and spinor fields. We shall begin with some properties of G_2.

7.1 The group G_2

It is well-known that the only normed real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions $\mathbb{O}$. The corresponding real algebra automorphism groups are 1, $\mathbb{Z}_2$, SO_3 and $G_2 \overset{\text{def}}{=} \mathrm{Aut}_{\mathbb{R}}\mathbb{O}$, and these are real Lie groups. In this section we will study the properties of G_2.

As a real representation of its automorphism group G_2, the space $\mathbb{O}$ splits into irreducibles as $\mathrm{Re}\mathbb{O} \oplus \mathrm{Im}\mathbb{O}$ where $\mathrm{Re}\mathbb{O} \cong \mathbb{R}$ and $\mathrm{Im}\mathbb{O}$ is the 7-dimensional space of imaginary octonions. Via the vector space isomorphism $\mathbb{O} \cong \mathbb{R}^8$ the octonions inherit a positive-definite inner product g and the above splitting is orthogonal. This can be written in terms of octonionic multiplication and conjugation (which negates the imaginary part) and the formula is the same as for the quaternionic version:

$$g(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x}) \ , \ x, y \in \mathbb{O} \ .$$

Since G_2 acts by automorphisms it preserves g, so $G_2 < SO_8$. The orthogonality of the splitting tells us that G_2 preserves the restriction of g to $\mathrm{Im}\mathbb{O}$ and we furthermore find that $G_2 < SO_7$. From this perspective $\mathrm{Im}\mathbb{O}$ is the standard 7-dimensional real representation of SO_7, and it remains irreducible when we restrict to G_2.

At this point the reader is referred to Appendix A, which contains our notation for octonions.

Definition 7.1.1. *Let the **canonical form** $\phi_0 : \mathrm{Im}\mathbb{O} \times \mathrm{Im}\mathbb{O} \times \mathrm{Im}\mathbb{O} \to \mathbb{R}$ be*

$$\phi_0(x,y,z) \stackrel{\text{def}}{=} -\frac{1}{6}\mathrm{Re}\big[(xy)z\big] ,$$

i.e.

$$\phi_0 = -\frac{1}{6}\mathrm{Re}\underset{\rightarrow}{\cdot} .$$

The $-1/6$ will be convenient later. Using the multiplication table in Appendix A it may be shown that $\mathrm{Re}\big[(xy)z\big] = \mathrm{Re}\big[x(yz)\big] = \mathrm{Re}\underset{\longleftarrow}{xyz}$, so $\phi_0 = -(1/6)\mathrm{Re}\underset{\leftarrow}{\cdot}$ as well.

Proposition 7.1.2. *We have*

1. *ϕ_0 is totally skew,*

2. *The stabiliser of ϕ_0 is G_2.*

Proof. 1.

$$\phi_0(x,x,z) = -\frac{1}{6}\mathrm{Re}\big[(xx)z\big] = -\frac{1}{6}(xx)\mathrm{Re}z = 0 ,$$

$$\phi_0(x,y,x) = -\frac{1}{6}\mathrm{Re}\big[(xy)y\big] = -\frac{1}{6}\mathrm{Re}\big[x(yy)\big] = -\frac{1}{6}(yy)\mathrm{Re}x = 0 ,$$

so ϕ_0 is skew by linearity and may be considered a map $\Lambda^3\mathrm{Im}\mathbb{O} \to \mathbb{R}$.

2. See [Sal89].

□

Put $V = \mathrm{Im}\mathbb{O}$, and then G_2 is the stabiliser of $\phi_0 \in \Lambda^3 V$. By evaluating ϕ_0 on basis elements, for example

$$\phi_0(1 \wedge 2 \wedge 3) = -\frac{1}{6}\mathrm{Re}\big[(12)3\big] = \frac{1}{6} ,$$

we find that

$$\begin{aligned}\phi_0 = {} & 1 \wedge 2 \wedge 3 + 1 \wedge 4 \wedge 5 - 1 \wedge 6 \wedge 7 + 2 \wedge 4 \wedge 6 \\ & + 2 \wedge 5 \wedge 7 + 3 \wedge 4 \wedge 7 - 3 \wedge 5 \wedge 6 .\end{aligned}$$

It will also be very useful to note that

$$\begin{aligned}*\phi_0 = {} & 4 \wedge 5 \wedge 6 \wedge 7 + 2 \wedge 3 \wedge 6 \wedge 7 - 2 \wedge 3 \wedge 4 \wedge 5 + 1 \wedge 3 \wedge 5 \wedge 7 \\ & + 1 \wedge 3 \wedge 4 \wedge 6 + 1 \wedge 2 \wedge 5 \wedge 6 - 1 \wedge 2 \wedge 4 \wedge 7 .\end{aligned}$$

Denote by $\mathfrak{g}_2$ the Lie algebra of G_2, so $\mathfrak{g}_2 \subset \mathfrak{so}_7$. A basis for $\mathfrak{so}_7$ is $1 \wedge 2, 1 \wedge 3, \ldots, 6 \wedge 7$ and the Lie algebra action on $\Lambda^3 V$ is given by

$$(a \wedge b)(1 \wedge 2 \wedge 3) = (a \wedge b)(1) \wedge 2 \wedge 3 + 1 \wedge (a \wedge b)(2) \wedge 3 + 1 \wedge 2 \wedge (a \wedge b)(3)$$

which gives for example

$$(1 \wedge 2)(\phi_0) = \frac{1}{2}\big[-2 \wedge 4 \wedge 5 + 2 \wedge 6 \wedge 7 + 1 \wedge 4 \wedge 6 + 2 \wedge 5 \wedge 7\big] .$$

If we do this for all elements of our basis for $\mathfrak{so}_7$ we see that

$$\begin{aligned}
1 \wedge 2(\phi_0) &= 4 \wedge 7(\phi_0) = -5 \wedge 6(\phi_0)\ , \\
1 \wedge 3(\phi_0) &= -4 \wedge 6(\phi_0) = -5 \wedge 7(\phi_0)\ , \\
1 \wedge 4(\phi_0) &= -2 \wedge 7(\phi_0) = 3 \wedge 6(\phi_0)\ , \\
1 \wedge 5(\phi_0) &= 3 \wedge 7(\phi_0) = 2 \wedge 6(\phi_0)\ , \\
1 \wedge 6(\phi_0) &= -2 \wedge 5(\phi_0) = -3 \wedge 4(\phi_0)\ , \\
1 \wedge 7(\phi_0) &= 2 \wedge 4(\phi_0) = -3 \wedge 5(\phi_0)\ , \\
2 \wedge 3(\phi_0) &= 4 \wedge 5(\phi_0) = -6 \wedge 7(\phi_0)\ .
\end{aligned}$$

It is clear that $\mathfrak{g}_2$ can be identified as the subalgebra of $\mathfrak{so}_7$ which sends ϕ_0 to zero; a basis of $\mathfrak{g}_2$ can be read off as

$$\begin{gathered}
1 \wedge 2 - 4 \wedge 7\ ,\ 1 \wedge 2 + 5 \wedge 6\ ,\ 1 \wedge 3 + 4 \wedge 6\ ,\ 1 \wedge 3 + 5 \wedge 7\ , \\
1 \wedge 4 + 2 \wedge 7\ ,\ 1 \wedge 4 - 3 \wedge 6\ ,\ 1 \wedge 5 - 2 \wedge 6\ ,\ 1 \wedge 5 - 3 \wedge 7\ , \\
1 \wedge 6 + 2 \wedge 5\ ,\ 1 \wedge 6 + 3 \wedge 4\ ,\ 1 \wedge 7 - 2 \wedge 4\ ,\ 1 \wedge 7 + 3 \wedge 5\ , \\
2 \wedge 3 - 4 \wedge 5\ ,\ 2 \wedge 3 + 6 \wedge 7\ .
\end{gathered}$$

We can see $\dim_{\mathbb{R}} \mathfrak{g}_2 = 14$ and so G_2 is also 14-dimensional. We note at this point the well-known fact that G_2 is simply connected.
So far we have worked exclusively in the frame $1, \ldots, 7$ of V.

Definition 7.1.3. *A **Cayley frame** of V is a frame obtained from $1, \ldots, 7$ by acting with an element of G_2.*

The Cayley frames form a G_2-subtorsor of the torsor of frames of V. Of course they are orthonormal and as a basis (along with 1) of $\mathbb{O}$ the multiplication table in Appendix A would appear merely permuted. The above calculations would remain nearly the same, whereas in an arbitrary orthonormal frame they may become very complicated.

Proposition 7.1.4. *Upon restriction to $G_2 \subset Spin_7$ we have the following decompositions into irreducibles.*

1. $$\Lambda^2 V = \mathfrak{g}_2 \oplus V\ .$$

2. $$\Lambda^3 V = \mathbb{R}\phi_0 \oplus V \oplus S^2_0 V\ .$$

Proof. These are shown in [FG82]. □

From the basis for $\mathfrak{g}_2$ above, we can easily produce a basis for the complement $V \subset \Lambda^2 V$:

$$\begin{gathered}
1 \wedge 2 + 4 \wedge 7 - 5 \wedge 6\ ,\ 1 \wedge 3 - 4 \wedge 6 - 5 \wedge 7\ ,\ 1 \wedge 4 - 2 \wedge 7 + 3 \wedge 6\ , \\
1 \wedge 5 + 2 \wedge 6 + 3 \wedge 7\ ,\ 1 \wedge 6 - 2 \wedge 5 - 3 \wedge 4\ ,\ 1 \wedge 7 + 2 \wedge 4 - 3 \wedge 5\ , \\
2 \wedge 3 + 4 \wedge 5 - 6 \wedge 7\ .
\end{gathered}$$

7.2 G_2-structures

Definition 7.2.1. *A G_2-**structure** B_{G_2} on $\mathcal{M}^7$ is a G_2-subbundle of the bundle of frames $GL(\mathcal{M})$ of $\mathcal{M}$. We call the elements (and local sections) of B_{G_2} the **Cayley frames** of $\mathcal{M}$.*

A G_2-structure B_{G_2} brings with it a rich geometry. Since G_2 lies inside SO_7, the natural enlargement

$$B_{G_2} \times_{G_2} SO_7 \subset GL(\mathcal{M})$$

is a Riemannian structure, i.e. it defines a Riemannian metric g on $\mathcal{M}$. Furthermore[1]

Theorem 7.2.2. *([LM89]) A smooth manifold $\mathcal{M}^7$ admits a G_2-structure if and only if it is spin.*

We can say slightly more than this; since G_2 is simply connected, its preimage under the standard double covering $Spin_7 \to SO_7$ is another copy of G_2, which we call $\tilde{G}_2$. Then if B_{G_2} is a G_2-structure,

$$B_{G_2} \times_{\tilde{G}_2} Spin_7$$

is a spin structure on $\mathcal{M}$. However, a spin structure alone *does not* pick out a G_2-structure. This can be understood by observing that a spin structure carries no local information, whereas a G_2-structure does (it is a *structure* in the true sense of the word).
We have explained that for V the standard 7-dimensional real representation of G_2, there is an invariant element $\phi_0 \in \Lambda^3 V$. Since $\Lambda^3 T\mathcal{M} \cong B_{G_2} \times_{G_2} \Lambda^3 V$ is an isomorphism of associated bundles we get a canonical 3-form ϕ on $\mathcal{M}$. By *2* of Lemma 7.1.2, the stabliser of ϕ_0 is G_2 and we can recover B_{G_2} from ϕ. In a Cayley frame $1, \ldots, 7$

$$\begin{aligned} \phi = {} & 1 \wedge 2 \wedge 3 + 1 \wedge 4 \wedge 5 - 1 \wedge 6 \wedge 7 + 2 \wedge 4 \wedge 6 \\ & + 2 \wedge 5 \wedge 7 + 3 \wedge 4 \wedge 7 - 3 \wedge 5 \wedge 6 \ . \end{aligned}$$

Denote by ∇ the Levi-Civita covariant derivative associated to the metric g on $\mathcal{M}$. In the paper of Fernández and Gray [FG82], G_2-structures are divided into 16 types based on splitting $\nabla\phi$ into components. We outline this procedure here.
The derivative $\nabla\phi$ is a section of the associated bundle

$$T\mathcal{M} \otimes \Lambda^3 T\mathcal{M} = B_{G_2} \times_{G_2} (V \otimes \Lambda^3 V)$$

which splits into irreducible subbundles corresponding to the irreducible summands of the G_2-representation $V \otimes \Lambda^3 V$. As an SO_7-representation we have

$$V \otimes \Lambda^3 V = \Lambda^4 V \oplus \Lambda^2 V \oplus \Lambda^{3,1} V$$

where the final summand is defined merely to be the orthogonal complement of the first two.

[1] This obstruction was originally proved in the context of cross products by Gray [Gra69].

To prove this we must write down all the necessary equivariant maps which split the exact sequences resulting from the projections onto the different summands. We have done this and omit the proofs. Now, we can apply these to an element of the right-hand side to get the formula

$$\begin{aligned} w \otimes x \wedge y \wedge z = \frac{1}{4}\big[& w \otimes x \wedge y \wedge z - x \otimes y \wedge z \wedge w \\ & + y \otimes z \wedge w \wedge x - z \otimes w \wedge x \wedge y \big] \\ & + \frac{1}{5} g \wedge \big[g(w,x) y \wedge z + g(w,y) z \wedge x + g(w,z) x \wedge y \big] \\ & + \big[w \otimes x \wedge y \wedge z \big]_{\Lambda^{3,1}V} \ . \end{aligned}$$

Now when we return to the bundle picture, we can use this decomposition on $\nabla\phi$ and this gives us

$$\nabla\phi = d\phi + \frac{3}{5} g \wedge \delta\phi + \mathcal{T}\phi$$

where δ is the coderivative, $g \wedge \delta\phi = \sum_{\mathfrak{i}=1}^{7} \mathfrak{i} \otimes (\mathfrak{i} \wedge \delta\phi)$ and $\mathcal{T}$ is defined to be the projection onto the third summand, known as the *twistor operator of forms.*

The Hodge star $*$ gives us an isomorphism $\Lambda^4 V \cong \Lambda^3 V$, and we also have the splittings given in Proposition 7.1.4

$$\Lambda^2 V = \mathfrak{g}_2 \oplus V \ ,$$

$$\Lambda^3 V = \mathbb{R}\phi_0 \oplus V \oplus S^2_0 V \ .$$

Furthermore $\Lambda^{3,1}V$ splits as

$$\Lambda^{3,1}V = V \oplus \mathfrak{g}_2 \oplus S^2_0 V \oplus C^{64} \oplus S^3_0 V$$

where C^{64} is a 64-dimensional irreducible. Thus, as a representation of G_2,

$$V \otimes \Lambda^3 V = (\mathbb{R}\phi_0 \oplus V \oplus S^2_0 V) \oplus (\mathfrak{g}_2 \oplus V) \oplus (V \oplus \mathfrak{g}_2 \oplus S^2_0 V \oplus C^{64} \oplus S^3_0 V) \ .$$

The resulting decomposition of a single element of $V \otimes \Lambda^3 V$ is very complicated to write down, and so we omit it. However, when we apply this decomposition to $\nabla\phi$ we find

$$\begin{aligned} \nabla\phi = g(*d\phi, \phi) * \phi + 24 * \big[* \phi(d\phi(\phi)) \big] + \big[\nabla\phi\big]_{S^2_0 V} \\ + \frac{3}{5} g \wedge \delta\phi_{\mathfrak{g}_2} + \frac{36}{5} g \wedge \phi(\phi(\delta\phi)) + \mathcal{T}\phi \end{aligned}$$

where we have no elegant expression for the $S^2_0 V$-component of $d\phi$, the $\mathfrak{g}_2$-component of the coderivative $\delta\phi$ is given by restriction $\Lambda^2 V \cong \mathfrak{so}_7 \supset \mathfrak{g}_2$, and $\mathcal{T}\phi$ is the $\Lambda^{3,1}V$ part all in one piece.

Lemma 7.2.3. *The V-components of $d\phi$ and $\delta\phi$ are proportional; more precisely*

$$d\phi(\phi) = \frac{9}{20}\phi(\delta\phi) \ .$$

Proof. Consider the left-hand side. Looking at just one term,

$$d(1 \wedge 2 \wedge 3)(\phi) = (d1 \wedge 2 \wedge 3 - 1 \wedge d2 \wedge 3 + 1 \wedge 2 \wedge d3)(\phi) \ .$$

When we contract $d1 \wedge 2 \wedge 3$ with ϕ, the only term of ϕ which can make a contribution is $1 \wedge 2 \wedge 3$ because it is the only one which has both 2 and 3 in it:

$$(d1 \wedge 2 \wedge 3)(\phi) = (d1 \wedge 2 \wedge 3)(1 \wedge 2 \wedge 3) \ .$$

Furthermore, only the part of $d1$ containing 1 can contribute, and if

$$d1 = \sum_{\mathrm{i},\mathrm{j}=1}^{7} (d1)^{\mathrm{ij}}\mathrm{i} \wedge \mathrm{j}$$

then the part containing 1 is

$$\sum_{\mathrm{j}=1}^{7} (d1)^{1\mathrm{j}} 1 \wedge \mathrm{j} + \sum_{\mathrm{i}=1}^{7} (d1)^{\mathrm{i}1} \mathrm{i} \wedge 1 = 2\sum_{\mathrm{j}=1}^{7} (d1)^{1\mathrm{j}} 1 \wedge \mathrm{j} \ .$$

On the other hand

$$d1(1) = \sum_{\mathrm{j}=1}^{7} (d1)^{1\mathrm{j}}\mathrm{j}$$

so the part of $d1$ containing 1 is $2 1 \wedge d1(1)$. Then

$$(d1 \wedge 2 \wedge 3)(1 \wedge 2 \wedge 3) = 2 1 \wedge 2 \wedge 3 \wedge d1(1)(\phi) = \frac{1}{3} d1(1) \ .$$

Thus

$$d(1 \wedge 2 \wedge 3)(\phi) = \frac{1}{3}\big[d1(1) + d2(2) + d3(3)\big] \ .$$

Since each of the i's appears three times in the expression for ϕ, we get

$$d\phi(\phi) = \sum_{\mathrm{i}=1}^{7} d\mathrm{i}(\mathrm{i}) \ .$$

Now to calculate $\phi(\delta\phi)$, observe that

$$\begin{aligned}\phi(\delta\phi) &= \frac{1}{6} * (\delta\phi \wedge *\phi) \\ &= -\frac{1}{18} * (*d * \phi \wedge *\phi) \\ &= -\frac{1}{18} * (*\phi \wedge *d * \phi) \\ &= -\frac{20}{3}(d * \phi)(*\phi)\end{aligned}$$

using Lemma B.0.17 twice and the expression $\delta\phi = -\frac{1}{3} * d * \phi$. We can apply the same method to this expression that we used above for $d\phi(\phi)$ to show that

$$(d * \phi)(*\phi) = -\frac{1}{3}\sum_{\mathfrak{i}=1}^{7} d\mathfrak{i}(\mathfrak{i}) \ .$$

The result follows. □

Theorem D.0.39 tells us that $\nabla\phi$ is identified with the torsion of the G_2-structure, and this takes values in the bundle associated to $\mathfrak{g}_2^\perp \otimes V$, where $\mathfrak{g}_2^\perp$ is the orthogonal complement of $\mathfrak{g}_2 \subset \mathfrak{so}_7$. Proposition 7.1.4 shows that $\mathfrak{g}_2^\perp = V$. According to [Sal89],

$$V \otimes V = \mathbb{R} \oplus V \oplus S_0^2 V \oplus \mathfrak{g}_2$$

and this means the torsion may only have $\mathbb{R}$, V, S_0^2V and $\mathfrak{g}_2$ parts. The components of $\nabla\phi$ corresponding to the irreducibles C^{64} and S_0^3V must vanish. David [Dav11] has shown that the V, S_0^2V and $\mathfrak{g}_2$ parts appearing in $\mathcal{T}\phi$ are proportional to those already appearing in the other terms, and therefore

Theorem 7.2.4. *([Dav11]) The canonical* 3*-form* ϕ *is conformal-Killing if and only if it is Killing.*

So there are four components of $\nabla\phi$ which are important. We can classify G_2-structures by which of these components are non-zero, and this gives us sixteen *form-classes*. Each form-class can be described with a differential equation that the canonical form must satisfy, and we tabulate these below. This was first achieved by Fernández and Gray [FG82], following the original method of Gray and Hervella [GH80].

Theorem 7.2.5. *([FG82]) The* ***form-classes*** *of* G_2*-structures are given by the following relations. The notation of the following table matches ours, and so differs considerably from the paper [FG82].*

Form-class	*Defining relations*
0	$\nabla\phi = 0$
$\mathbb{R}$	$\nabla\phi = d\phi = g(*d\phi,\phi) * \phi\ ,\ \delta\phi = 0\ \ ;\ \ \phi$ *Killing*
V	$g(*d\phi,\phi) = 0$ *and* $\left[\nabla\phi\right]_{S_0^2V} = 0$ *and* $\delta\phi_{\mathfrak{g}_2} = 0$
S_0^2V	$\delta\phi = 0$ *and* $g(*d\phi,\phi) = 0$
$\mathfrak{g}_2$	$d\phi = 0$
$\mathbb{R}\oplus V$	$\left[\nabla\phi\right]_{S_0^2V} = 0$ *and* $\delta\phi_{\mathfrak{g}_2} = 0$
$\mathbb{R}\oplus S_0^2V$	$\delta\phi = 0$
$\mathbb{R}\oplus\mathfrak{g}_2$	$\phi(\delta\phi) = 0$ *and* $\left[\nabla\phi\right]_{S_0^2V} = 0$
$V\oplus S_0^2V$	$g(*d\phi,\phi) = 0$ *and* $\delta\phi_{\mathfrak{g}_2} = 0$
$V\oplus\mathfrak{g}_2$	$g(*d\phi,\phi) = 0$ *and* $\left[\nabla\phi\right]_{S_0^2V} = 0$
$S_0^2V\oplus\mathfrak{g}_2$	$g(*d\phi,\phi) = 0$ *and* $\phi(\delta\phi) = 0$
$\mathbb{R}\oplus V\oplus S_0^2V$	$\delta\phi_{\mathfrak{g}_2} = 0$
$\mathbb{R}\oplus V\oplus\mathfrak{g}_2$	$\left[\nabla\phi\right]_{S_0^2V} = 0$
$\mathbb{R}\oplus S_0^2V\oplus\mathfrak{g}_2$	$\phi(\delta\phi) = 0$
$V\oplus S_0^2\oplus\mathfrak{g}_2$	$g(*d\phi,\phi) = 0$
$\mathbb{R}\oplus V\oplus S_0^2\oplus\mathfrak{g}_2$	*No relation*

Proof. Clear from decomposition. □

Theorem 7.2.6. *([CMS96]) The form-class $\mathbb{R}\oplus\mathfrak{g}_2$ does not occur.*

Proof. Suppose $\nabla\phi$ has type $\mathbb{R}\oplus\mathfrak{g}_2$. Then

$$d\phi = g(*d\phi,\phi) * \phi$$

and we write $g(*d\phi,\phi) = f$ for convenience. Differentiating again gives

$$df \wedge *\phi = -f d * \phi$$

which is an equation of 5-forms. Apply the Hodge star to the left-hand side:

$$*(df \wedge *\phi) = 3\iota_{df}\phi$$

using Lemma B.0.17. This is a 2-form of type V. Apply the Hodge star to the other side:

$$*(-fd * \phi) = -f * d * \phi = 3f\delta\phi = 3f\delta\phi_{\mathfrak{g}_2}$$

which is a 2-form of type $\mathfrak{g}_2$. Since the two sides have different type, they must both be zero. The $\mathfrak{g}_2$-part therefore vanishes and $\nabla\phi$ must have type $\mathbb{R}$ or 0. □

Theorem 7.2.7. *([FG82, FI86, Cab96, Joy96b, CMS96]) All the other form-classes of G_2-structures occur, and moreover they occur with compact examples.*

The G_2-structures of type $\mathbb{R}$ are known as *nearly-parallel* or sometimes *weak holonomy* G_2 (in the sense of Gray [Gra71]). These have been studied in greater detail than the other classes (other than, of course, the parallel class), for example in [FKMS97]. Such structures also occur in the famous classification of Bär [Bär93], the most common example being that present on the sphere S^7. Submanifolds of S^7 have been studied (in the context we will see in Section 9.1) by Lotay [Lot10].

A natural question to ask is, which of the classes are conformally invariant? This was answered by Fernández and Gray [FG82] and the answer is that a class is conformally invariant if it contains a V summand. The G_2-structures of type V are known as *locally conformally-parallel* because a particular local conformal change of metric (resulting from defining a new 3-form as a positive function times the old one) makes such a structure parallel.

Many of the other classes have analogues in the classification of almost-Hermitian structures given by Gray and Hervella [GH80]. The similarity is no surprise given the unifying perspective explained in Appendix C. It may be possible to formulate some deep questions about G_2-structures, such as an analogue of the Calabi conjecture, but this has not yet been achieved.

Note that a manifold with G_2-structure has a naturally-defined vector field, given by its V-part. Specifically,

Definition 7.2.8. *The* ***canonical vector field*** *θ on a manifold $\mathcal{M}$ with G_2-structure is the projection of $*d\phi$ onto $T\mathcal{M}$:*

$$\theta \stackrel{\text{def}}{=} -24 d\phi(\phi) \ .$$

Using Lemma 7.2.3 we see that the projection of $\delta\phi$ onto $T\mathcal{M}$ is $\frac{10}{9}\theta$.

Lemma 7.2.9. *The V-part of $d\phi$ is*

$$\frac{1}{4}\theta \wedge \phi \ .$$

Proof. This follows easily from Lemma B.0.17. □

Definition 7.2.10. *The* ***canonical symmetric-traceless tensor field*** *T on a manifold $\mathcal{M}$ with G_2-structure is the projection of $*d\phi$ onto $S^2_0 T\mathcal{M}$, given by:*

$$(*d\phi)_{S^2_0 V} = T(\phi)$$

where T acts by its Lie algebra action, noting that S_0^2V is a Lie subalgebra of $\mathfrak{gl}(V)$. It is easy to show that the S_0^2V-part of $d\phi$ is given by

$$(d\phi)_{S_0^2V} = -T(*\phi) \ .$$

There is also a naturally-defined section of the adjoint bundle $B_{G_2} \times_{Ad} \mathfrak{g}_2$, but we shall not use it in this thesis.

7.3 The spin representation of G_2

We know that G_2 lives inside SO_7 and that it is simply connected, so its preimage under the standard double covering $Spin_7 \to SO_7$ is another copy of G_2, called $\tilde{G}_2$. Denote by Δ the real 8-dimensional representation of $Spin_7$. What happens when we restrict this representation to $\tilde{G}_2$?

The real representation Δ has a nice description. Put $V = \mathrm{Im}\mathbb{O}$ with its usual Cayley frame $1, \ldots, 7$. The members of this frame are generators for the real Clifford algebra $C\ell_7$, which is non-canonically isomorphic to the sum of matrix algebras $\mathbb{R}(8) \oplus \mathbb{R}(8)$. Define two maps

$$\begin{aligned}
\mathfrak{L} : V \to \mathrm{End}\mathbb{O} : u \to \mathfrak{L}(u) : x \to \mathfrak{L}(u)x = ux \ , \\
\mathfrak{R} : V \to \mathrm{End}\mathbb{O} : u \to \mathfrak{R}(u) : x \to \mathfrak{R}(u)x = xu
\end{aligned}$$

where $\mathrm{End}\mathbb{O}$ denotes the vector space endomorphisms (not algebra endomorphisms) of $\mathbb{O}$. Then

$$\mathfrak{L}(u)\mathfrak{L}(u)x = u(ux) = (uu)x = -g(u,u)x$$

by the alternative property of $\mathbb{O}$, and similarly for $\mathfrak{R}$. Thus $\mathfrak{L}$ and $\mathfrak{R}$ extend to homomorphisms

$$\mathfrak{L}, \mathfrak{R} : C\ell_7 \to \mathrm{End}\mathbb{O}$$

which are easily seen to be the two irreducible real Clifford representations. Conjugation $x \to \bar{x}$ on $\mathbb{O}$ gives

$$\overline{(-xu)} = -\bar{u}\bar{x} = u\bar{x}$$

so the representation defined by $\mathfrak{R}'(u)x = -xu$, which also extends to $C\ell_7$, is equivalent to $\mathfrak{L}$. Similarly $\mathfrak{L}'(u)x = -ux$ is equivalent to $\mathfrak{R}$. We choose Δ to be the one where the volume element ω acts as multiplication by $+1$, and the multiplication table in Appendix A tells us

$$\mathfrak{L}(\omega)x = (\underleftarrow{1234567})x = -x \ .$$

Definition 7.3.1. *The **real Clifford representation** in seven dimensions is $(\mathfrak{R}, \Delta)$ where $\Delta = \mathbb{O}$ and $\mathfrak{R}$ is given on $V \subset C\ell_7$ by $\mathfrak{R}(u)x = xu$. We shall often use the standard notation $u \cdot x \stackrel{\mathrm{def}}{=} \mathfrak{R}(u)x$.*

When we restrict $\mathfrak{R}$ to the even part $C\ell_7^0 \subset C\ell_7$, $\mathfrak{L}$ and $\mathfrak{R}$ become equivalent by conjugation:

$$\begin{aligned}
\overline{\mathfrak{L}(u_1u_2)x} &= \overline{\mathfrak{L}(u_1)\mathfrak{L}(u_2)x} = \overline{u_1(u_2x)} = (\overline{u_2x})\bar{u_1} \\
&= (\bar{x}\bar{u_2})\bar{u_1} = (\bar{x}u_2)u_1 = \mathfrak{R}(u_1)\mathfrak{R}(u_2)\bar{x} = \mathfrak{R}(u_1u_2)\bar{x} \ .
\end{aligned}$$

From the representation theory of spinors we have that the restriction of $\mathfrak{R}$ to $Spin_7 \subset C\ell_7$, to get the *spin representation*, also denoted Δ, leaves Δ irreducible. Now restrict to $\tilde{G_2} \subset Spin_7$ in the way prescribed above to get an action of $\tilde{G_2}$ on $\Delta = \mathbb{O}$.

Lemma 7.3.2. *The diagram*

$$\begin{tikzcd} \tilde{G_2} \arrow[dd] \arrow[dr, "\mathfrak{R}"] & \\ & \mathrm{End}\Delta \\ G_2 \arrow[ur] & \end{tikzcd}$$

commutes.

Proof. Remember $\tilde{G_2} \xrightarrow{\sim} G_2$ is the restriction of the standard double cover $Spin_7 \to SO_7$ and $G_2 \to \mathrm{End}\mathbb{O}$ is the natural inclusion of real algebra automorphisms $G_2 = \mathrm{Aut}_{\mathbb{R}}\mathbb{O}$ into $\mathrm{End}\mathbb{O}$. It is easier to ask the same question for Lie algebras $\tilde{\mathfrak{g}_2}$ and $\mathfrak{g}_2$, since we have an explicit basis for $\mathfrak{g}_2$ and we know the isomorphism $\mathfrak{spin}_7 \xrightarrow{\sim} \mathfrak{so}_7$ induced by the standard double cover is given by (see Lemma 5.2.1)

$$\mathfrak{spin}_7 \ni \mathfrak{i}\mathfrak{j} \to 4\mathfrak{i} \wedge \mathfrak{j} \in \mathfrak{so}_7$$

for $\mathfrak{i}, \mathfrak{j}$ chosen from $\mathtt{1}, \ldots, \mathtt{7} \in V$. Now we can take our basis for $\mathfrak{g}_2$ and act each member on $x \in \mathbb{O}$ in the natural way, e.g.

$$(\mathtt{1} \wedge \mathtt{2} - \mathtt{4} \wedge \mathtt{7})x = \frac{1}{2}\left[g(\mathtt{1}, x)\mathtt{2} - g(\mathtt{2}, x)\mathtt{1} - g(\mathtt{4}, x)\mathtt{7} + g(\mathtt{7}, x)\mathtt{4}\right]$$

and also using $\mathfrak{R}$ (or $\mathfrak{L}$, which will give the same answer) to get

$$\frac{1}{4}\mathfrak{R}(\mathtt{1}\mathtt{2} - \mathtt{4}\mathtt{7})x = \frac{1}{4}\left[(x\mathtt{2})\mathtt{1} - (x\mathtt{7})\mathtt{4}\right] .$$

By substituting in the basis elements $1, \mathtt{1}, \ldots, \mathtt{7}$ of $\mathbb{O}$ in place of x, we can check that these two actions are equal. The same must hold for the diagram with Lie groups instead of algebras. $\square$

Theorem 7.3.3. *The restriction of the real spin representation $(\mathfrak{R}, \Delta)$ to $\tilde{G_2} \subset Spin_7$ has an invariant element φ_0, which we shall call the* ***canonical spinor****.*

Proof. $1 \in \mathbb{O}$ is invariant under $G_2 = \mathrm{Aut}_{\mathbb{R}}\mathbb{O}$. $\square$

We can say more; since the action of $\tilde{G_2}$ on $\Delta = \mathbb{O}$ is the same as that of G_2, we have

$$\Delta = \mathbb{R}\varphi_0 \oplus V \cdot \varphi_0$$

which is just our new way of writing $\mathbb{O} = \mathrm{Re}\mathbb{O} \oplus \mathrm{Im}\mathbb{O}$. In other words, a spinor can be thought of as an ordinary vector plus a part proportional to φ_0. This will make things very simple from now on.

Proposition 7.3.4. *We have*

1. *The action of 2-forms on φ_0 by Clifford multiplication yields a homomorphism $\Lambda^2 V \to V \cdot \varphi_0$ such that*

$$\mathfrak{g}_2 \to 0\ ,\ V \xrightarrow{\sim} V \cdot \varphi_0$$

where we have used the simple notation $\Lambda^2 V = \mathfrak{g}_2 \oplus V$ for the G_2-decomposition of 2-forms.

2. *The action of 3-forms on φ_0 by Clifford multiplication yields a homomorphism $\Lambda^3 V \to \Delta = \mathbb{R}\varphi_0 \oplus V \cdot \varphi_0$ such that*

$$\mathbb{R}\phi_0 \xrightarrow{\sim} \mathbb{R}\varphi_0 \ , \ V \xrightarrow{\sim} V \cdot \varphi_0 \ , \ S_0^2 V \to 0$$

where we have used the simple notation $\Lambda^3 V = \mathbb{R}\phi_0 \oplus V \oplus S_0^2 V$ for the G_2-decomposition of 3-forms.

Proof. 1. Obvious using the identification of $\mathfrak{spin}_7$ with $\Lambda^2 V$.

2. It is easy to check $\phi_0 \cdot \varphi_0 = 7\varphi_0$. A 3-form of type V is of the form $\iota_v * \phi_0$, and a calculation gives

$$(\iota_v * \phi_0) \cdot \varphi_0 = -v \cdot \varphi_0 \ .$$

A 3-form of type $S_0^2 V$ is of the form $(u \wedge \iota_v + v \wedge \iota_u)\phi_0$ where $g(u, v) = 0$ (this may be verified by showing that this element of $\Lambda^3 V$ is orthogonal to ϕ_0 and all elements of type V). A calculation gives

$$u \wedge \iota_v \phi_0 \cdot \varphi_0 = 4\phi_0(u \wedge v \wedge \ \cdot \ ) \cdot \varphi_0$$

which is skew in u, v. Thus $(u \wedge \iota_v + v \wedge \iota_u)\phi_0 \cdot \varphi_0 = 0$. The fact that $S_0^2 V \to 0$ also follows from Schur's Lemma without a calculation.

□

7.4 The spinorial classification

Note that representations of SO_7 are also representations of $Spin_7$, so V is acted upon by $Spin_7$ and hence $\tilde{G}_2$. This action is identical to that of $G_2 \subset SO_7$ via the isomorphism $\tilde{G}_2 \to G_2$. Using the G_2-invariant element $\varphi_0 \in \Delta$ we get a natural spinor field on $\mathcal{M}$, denoted φ. The real spinor bundle of $\mathcal{M}$ (corresponding to the spin structure specified by B_{G_2}) breaks into irreducibles

$$\begin{aligned} S\mathcal{M} &\overset{\text{def}}{=} B_{G_2} \times_{\tilde{G}_2} \Delta \\ &= B_{G_2} \times_{\tilde{G}_2} (\mathbb{R}\varphi_0 \oplus V) \\ &= (\mathcal{M} \times \mathbb{R}\varphi_0) \oplus T\mathcal{M} \end{aligned}$$

and φ is the constant section of $\mathcal{M} \times \mathbb{R}\varphi_0$ corresponding to φ_0. Note that Theorem D.0.39 tells us that the torsion of the G_2-structure can be identified with $\nabla\varphi$. The torsion[2] $\nabla\varphi$ is a function on $\mathcal{M}$ with values in the associated bundle

$$T\mathcal{M} \otimes S\mathcal{M} = B_{G_2} \times_{\tilde{G}_2} (V \otimes \Delta)$$

which splits into irreducible subbundles corresponding to the irreducible summands of the $\tilde{G}_2$-representation $V \otimes \Delta$. As a $Spin_7$-representation we have

$$V \otimes \Delta = \Delta \oplus \mathcal{C} \ ,$$

[2]See Appendix D for details on torsion.

where $\mathcal{C}$ is the kernel of Clifford multiplication and the orthogonal complement of Δ in $V \otimes \Delta$. As with the form case we omit the details of the equivariant maps which split the exact sequences resulting from the projections onto the two summands. Returning to the bundle picture, we can apply this decomposition to $\nabla\varphi$ and this gives us

$$\nabla\varphi = -\frac{1}{7} g \cdot \mathcal{D}\varphi + \mathcal{T}\varphi$$

where $\cdot$ is Clifford multiplication (i.e. $\mathfrak{R}$), so $g \cdot \mathcal{D}\varphi = \sum_{\mathfrak{i}=1}^{7} \mathfrak{i} \otimes (\mathfrak{i} \cdot \mathcal{D}\varphi)$, $\mathcal{D}$ is the Dirac operator and $\mathcal{T}$ is defined to be the projection onto the summand associated to $\mathcal{C}$, known as the *twistor operator of spinors.*

Now when we restrict our representation to $\tilde{G}_2 \subset Spin_7$, $\nabla\varphi$ will split into more pieces. Rather than continue with $\Delta \oplus \mathcal{C}$, note instead that, under $\tilde{G}_2$,

$$\begin{aligned} V \otimes \Delta &= V \otimes (\mathbb{R}\varphi_0 \oplus V \cdot \varphi_0) \\ &= (V \otimes \mathbb{R}\varphi_0) \oplus (V \otimes V \cdot \varphi_0) \\ &= (V \otimes \mathbb{R}\varphi_0) \oplus (\mathbb{R}g \cdot \varphi_0) \oplus (S_0^2 V \cdot \varphi_0) \oplus (\mathfrak{g}_2 \cdot \varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0) \end{aligned}$$

where we have used the decomposition $V \otimes V = \mathbb{R}g \oplus S_0^2 V \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2^{\perp}$ so that in the final four terms above, $\mathbb{R}g$, $S_0^2 V$, $\mathfrak{g}_2$ and $\mathfrak{g}_2^{\perp} \cong V$ should be considered as subspaces of $V \otimes V$ acting on φ_0 by Clifford multiplication via $u \otimes (v \cdot \varphi_0)$. The reason we continued to use $V \otimes \mathbb{R}\varphi_0$ and $\mathbb{R}g \cdot \varphi_0$ rather than just V and $\mathbb{R}$ is that we will now be able to see which summands correspond to Δ and $\mathcal{C}$ in the $Spin_7$ decomposition.

Lemma 7.4.1. *Using Clifford multiplication $V \otimes \Delta \to \Delta$,*

$$\begin{aligned} V \otimes \mathbb{R}\varphi_0 &\xrightarrow{\sim} V \cdot \varphi_0 \subset \Delta \ , \\ \mathbb{R}g \cdot \varphi_0 &\xrightarrow{\sim} \mathbb{R}\varphi_0 \subset \Delta \ , \\ S_0^2 V \cdot \varphi_0 &\to 0 \ , \\ \mathfrak{g}_2 \cdot \varphi_0 &\to 0 \ , \\ \mathfrak{g}_2^{\perp} \cdot \varphi_0 &\xrightarrow{\sim} V \cdot \varphi_0 \subset \Delta \ . \end{aligned}$$

Proof. Calculation using Proposition 7.3.4. □

This shows that

$$\Delta = \mathbb{R}g \cdot \varphi_0 \oplus \mathrm{Diag}\big[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\big]$$

and

$$\mathcal{C} = S_0^2 V \cdot \varphi_0 \oplus \mathfrak{g}_2 \cdot \varphi_0 \oplus \mathrm{Antidiag}\big[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\big]$$

where

$$\mathrm{Diag}\big[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\big] \cong \mathrm{Antidiag}\big[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\big] \cong V$$

as representations of G_2 or $\tilde{G}_2$ (implicitly using Lemma 7.3.2). In total

$$-\frac{1}{7} g \cdot \mathcal{D}\varphi = -\frac{1}{7} g(\mathcal{D}\varphi, \varphi) g \cdot \varphi + \big[-\frac{1}{7} g \cdot \mathcal{D}\varphi\big]_{\mathrm{Diag}\left[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\right]}$$

and

$$\mathcal{T}\varphi = \big[\mathcal{T}\varphi\big]_{S_0^2 V \cdot \varphi_0} + \big[\mathcal{T}\varphi\big]_{\mathfrak{g}_2 \cdot \varphi_0} + \big[\mathcal{T}\varphi\big]_{\mathrm{Antidiag}\left[(V \otimes \mathbb{R}\varphi_0) \oplus (\mathfrak{g}_2^{\perp} \cdot \varphi_0)\right]} \ .$$

Lemma 7.4.2. *The V-components (the diagonal and antidiagonal pieces respectively) of $-\frac{1}{7}g \cdot \mathcal{D}\varphi$ and $\mathcal{T}\varphi$ are related by*

$$\big[-\frac{1}{7}g\cdot\mathcal{D}\varphi\big]_{V\otimes\mathbb{R}\varphi_0} = -\big[\mathcal{T}\varphi\big]_{V\otimes\mathbb{R}\varphi_0}$$

and

$$\big[-\frac{1}{7}g\cdot\mathcal{D}\varphi\big]_{\mathfrak{g}_2^\perp\cdot\varphi_0} = \big[\mathcal{T}\varphi\big]_{\mathfrak{g}_2^\perp\cdot\varphi_0}\ .$$

Proof. The second statement follows from the first. The projection $V\otimes\Delta\to V\otimes\mathbb{R}\varphi_0$ is given by taking the inner product of the spinor-valued 1-form with φ_0, yielding a 1-form, i.e. an element of V. Thus

$$\begin{aligned}\big[\mathcal{T}\varphi\big]_{V\otimes\mathbb{R}\varphi_0} &= g(\mathcal{T}\varphi,\varphi)\otimes\varphi\\ &= g(\nabla\varphi+\frac{1}{7}g\cdot\mathcal{D}\varphi,\varphi)\otimes\varphi\\ &= g(\frac{1}{7}g\cdot\mathcal{D}\varphi,\varphi)\otimes\varphi\\ &= -\big[-\frac{1}{7}g\cdot\mathcal{D}\varphi\big]_{V\otimes\mathbb{R}\varphi_0}\end{aligned}$$

because $g(\nabla\varphi,\varphi)=0$ since φ has constant unit norm. □

From here on we shall use a subscript V rather than the big diagonal and antidiagonal subscripts. We can prove a spinorial analogue of Theorem 7.2.4 of David.

Theorem 7.4.3. *If the canonical spinor field of a G_2-structure φ is a conformal-Killing spinor $\mathcal{T}\varphi=0$ then furthermore φ is Killing.*

Proof. This is immediate from Lemma 7.4.2. □

Lemma 7.4.2 shows that $\nabla\varphi$ has four components of importance. We can write a new classification table for G_2-structures with defining relations in terms of the spinor field φ.

Theorem 7.4.4. *G_2-structures can be classified into **spin-classes** as follows.*

Spin-class	*Defining relations*
0	$\nabla\varphi=0$
$\mathbb{R}$	$\nabla\varphi=-\frac{1}{7}g(\mathcal{D}\varphi,\varphi)g\cdot\varphi$; φ *Killing*
V	$\nabla\varphi=\big[-\frac{1}{7}g\cdot\mathcal{D}\varphi\big]_V+\big[\mathcal{T}\varphi\big]_V$
S^2_0V	$\nabla\varphi=\big[\mathcal{T}\varphi\big]_{S^2_0V\cdot\varphi_0}$; $\mathcal{D}\varphi=0$
$\mathfrak{g}_2$	$\nabla\varphi=\big[\mathcal{T}\varphi\big]_{\mathfrak{g}_2\cdot\varphi_0}$; $\mathcal{D}\varphi=0$

We do not complete the table, because without nice expressions for $\big[\mathcal{T}\varphi\big]_{S^2_0V\cdot\varphi_0}$, $\big[\mathcal{T}\varphi\big]_{\mathfrak{g}_2\cdot\varphi_0}$ etc., it will not be any more illuminating. The other classes are sums of the above ones, as in Theorem 7.2.5.

Proof. Clear from decomposition. □

The above theorem leads us to the following question: does the spin-class of a G_2-structure equal its form-class? To answer this we need to understand how to relate spinors and vectors in a useful way.

7.5 The squaring map

It is often said that spinors provide a refinement of forms. In precise terms

Definition 7.5.1. *We can define a homomorphism, called the **squaring map***

$$\Delta \otimes \Delta \to \Lambda V$$

which is $Spin_7$-equivariant, i.e. it is a homomorphism of representations. We can consider this map to be a sum of maps $\Delta \otimes \Delta \to \Lambda^k V$ for $k = 0, \ldots, 7$. These summands are given by

$$\varphi_1 \otimes \varphi_2 \to g(\mathfrak{R}(\cdot)\varphi_1, \varphi_2) \in \Lambda^k V$$

which acts on k-forms:

$$g(\mathfrak{R}(\cdot)\varphi_1, \varphi_2)(v_1 \wedge \ldots \wedge v_k) = g(\mathfrak{R}(v_1 \wedge \ldots \wedge v_k)\varphi_1, \varphi_2) \in \mathbb{R}$$

where $\mathfrak{R}$ is the Clifford representation as in Definition 7.3.1.

In particular we can 'square' φ_0 to a k-form. Assume $v_1, \ldots, v_k$ are mutually orthogonal, so that

$$\begin{aligned} g(v_1 \wedge \ldots \wedge v_k \cdot \varphi_0, \varphi_0) &= g(v_1 \cdot \ldots \cdot v_k \cdot \varphi_0, \varphi_0) \\ &= (-1)^k g(\varphi_0, v_k \cdot \ldots \cdot v_1 \cdot \varphi_0) \\ &= (-1)k(-1)^{k(k-1)/2} g(\varphi_0, v_1 \cdot \ldots \cdot v_k \cdot \varphi_0) \\ &= (-1)^{k(k+1)/2} g(v_1 \cdot \ldots \cdot v_k \cdot \varphi_0, \varphi_0) \end{aligned}$$

using the symmetry and invariance of g. So we only get a non-zero square when $k(k+1)/2$ is even, which in dimension 7 means $k = 0, 3, 4, 7$.

Proposition 7.5.2. *The square of φ_0 vanishes for $k = 1, 2, 5, 6$, and in the other cases*

$$\begin{aligned} k = 0 &: g(\varphi_0, \varphi_0) = 1 \\ k = 3 &: g(\mathfrak{R}(\cdot)\varphi_0, \varphi_0) = 6\phi_0 \\ k = 4 &: g(\mathfrak{R}(\cdot)\varphi_0, \varphi_0) = 24 * \phi_0 \\ k = 7 &: g(\mathfrak{R}(\cdot)\varphi_0, \varphi_0) = 7!\omega_7 \end{aligned}$$

where ω_7 is the volume form in eight dimensions. This allows us to write

$$\varphi_0 \otimes \varphi_0 \to 1 + 3!\phi_0 + 4! * \phi_0 + 7!\omega_7 \ .$$

Proof. The first formula is obvious.

$$\begin{aligned} g(\mathfrak{R}(x \wedge y \wedge z)\varphi_0, \varphi_0) &= g(\underrightarrow{zyx}, 1) \\ &= \mathrm{Re}(\underrightarrow{zyx}) \\ &= -6\phi_0(z \wedge y \wedge x) \\ &= 6\phi_0(x \wedge y \wedge z) \ . \end{aligned}$$

The third formula is similar using $*\phi_0 = \mathrm{Re}(\underleftarrow{\cdot})/24$, and

$$g(\mathfrak{R}(\mathfrak{1} \wedge \ldots \wedge \mathfrak{7})\varphi_0, \varphi_0) = \mathrm{Re}(\underrightarrow{\mathfrak{7} \cdots \mathfrak{1}}) = \mathrm{Re}(1) = 1 \ .$$

∎

This means the only interesting case is when $k = 3$ ($k = 4$ is equivalent). It will be useful for us to understand the map $\Delta \otimes \Delta \to \Lambda^3 V$ in more detail. We have

$$\begin{aligned} \Delta \otimes \Delta &= (\mathbb{R}\varphi_0 \oplus V \cdot \varphi_0) \otimes (\mathbb{R}\varphi_0 \oplus V \cdot \varphi_0) \\ &= (\mathbb{R}\varphi_0 \otimes \varphi_0) \oplus (\mathbb{R}\varphi_0 \otimes V \cdot \varphi_0) \oplus (V \cdot \varphi_0 \otimes \mathbb{R}\varphi_0) \oplus (V \cdot \varphi_0 \otimes V \cdot \varphi_0) \\ &= (\mathbb{R}\varphi_0 \otimes \varphi_0) \oplus (\mathbb{R}\varphi_0 \otimes V \cdot \varphi_0) \oplus (V \cdot \varphi_0 \otimes \mathbb{R}\varphi_0) \oplus (\mathbb{R}g \oplus S^2_0 V \oplus \mathfrak{g}_2 \oplus V) \cdot \varphi_0 \otimes \varphi_0 \end{aligned}$$

where the last four summands are understood to act on $\varphi_0 \otimes \varphi_0$ as $u \cdot \varphi_0 \otimes v \cdot \varphi_0$. Also

$$\Lambda^3 V = \mathbb{R}\phi_0 \oplus V \oplus S^2_0 V$$

where we remember that these are spaces of 3-forms to keep the notation simple.

Proposition 7.5.3. *Under the map $\Delta \otimes \Delta \to \Lambda^3 V$ defined above,*

1. *$\mathbb{R}\varphi_0 \otimes \varphi_0$ and $\mathbb{R}g \cdot \varphi_0 \otimes \varphi_0$ map to $\mathbb{R}\phi_0 \subset \Lambda^3 V$,*

2. *$\mathbb{R}\varphi_0 \otimes V \cdot \varphi_0$ and $V \cdot \varphi_0 \otimes \mathbb{R}\varphi_0$ map to $V \subset \Lambda^3 V$,*

3. *$S^2_0 V \cdot \varphi_0 \otimes \varphi_0$ maps to $S^2_0 V \subset \Lambda^3 V$,*

4. *$\mathfrak{g}_2 \cdot \varphi_0 \otimes \varphi_0$ and $V \cdot \varphi_0 \otimes \varphi_0$ map to 0.*

Proof. 1. We showed that $\mathbb{R}\varphi_0 \otimes \varphi_0$ maps to $\mathbb{R}\phi_0$ in the previous proposition. Also

$$g \cdot \varphi_0 \otimes \varphi_0 = \sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} \mathfrak{i} \cdot \varphi_0 \otimes \mathfrak{i} \cdot \varphi_0 \to \sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} g(\mathfrak{R}(\cdot)\mathfrak{i} \cdot \varphi_0, \mathfrak{i} \cdot \varphi_0)$$

which is a 3-form whose action on $x \wedge y \wedge z$ is

$$\sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{i} \cdot \varphi_0, \mathfrak{i} \cdot \varphi_0) = -\sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} g(\mathfrak{R}(\mathfrak{i})\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(\mathfrak{i})\varphi_0, \varphi_0) \ .$$

By Lemma B.0.22 we have

$$\sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} \mathfrak{R}(\mathfrak{i})\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(\mathfrak{i}) = \mathfrak{R}(x \wedge y \wedge z)$$

and so

$$g \cdot \varphi_0 \otimes \varphi_0 \to -g(\mathfrak{R}(\cdot)\varphi_0, \varphi_0) = -6\phi_0 \ .$$

2. An element of $\mathbb{R}\varphi_0 \otimes V \cdot \varphi_0$ is $\varphi_0 \otimes v \cdot \varphi_0$ and the image of this element is a 3-form whose action on $x \wedge y \wedge z$ is

$$g(\mathfrak{R}(x \wedge y \wedge z)\varphi_0, \mathfrak{R}(v)\varphi_0) = -g(\mathfrak{R}(v)\mathfrak{R}(x \wedge y \wedge z)\varphi_0, \varphi_0)$$

and $\mathfrak{R}(v)\mathfrak{R}(x \wedge y \wedge z)$ can be written as the action of a 4-form and some 2-form pieces. However, we know the square of φ_0 vanishes when $k = 2$, which leaves only the 4-form piece. Thus

$$\varphi_0 \otimes v \cdot \varphi_0 \to -g(\mathfrak{R}(v \wedge \ \cdot \ )\varphi_0, \varphi_0) = -24\iota_v * \phi_0$$

which of course is of type V. Similarly for $V \cdot \varphi_0 \otimes \mathbb{R}\varphi_0$.

3. An element of $S^2_0 V \cdot \varphi_0 \otimes \varphi_0$ is $u \cdot \varphi_0 \vee v \cdot \varphi_0$ where $\vee$ is symmetric wedge and we can assume u and v are orthogonal in V, so $g(u, v) = 0$. This subspace $S^2_0 V \cdot \varphi_0 \otimes \varphi_0 \subset \Delta \otimes \Delta$ is generated by such elements. The image of this element is a 3-form whose action on $x \wedge y \wedge z$ is

$$\frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(u)\varphi_0, \mathfrak{R}(v)\varphi_0) + \frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(v)\varphi_0, \mathfrak{R}(u)\varphi_0) \ .$$

Using Lemma B.0.22,

$$\begin{aligned} -\frac{1}{6} g(\mathfrak{R}(v)\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(u)\varphi_0, \varphi_0) &= g(v, x)\phi_0(y \wedge z \wedge u) \\ &- g(v, y)\phi_0(x \wedge z \wedge u) + g(v, z)\phi_0(x \wedge y \wedge u) + g(z, u)\phi_0(v \wedge x \wedge y) \\ &- g(y, u)\phi_0(v \wedge x \wedge z) + g(x, u)\phi_0(v \wedge y \wedge z) \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(u)\varphi_0, \mathfrak{R}(v)\varphi_0) + \frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(v)\varphi_0, \mathfrak{R}(u)\varphi_0) \\ &\quad = 3\big[g(v, x)\phi_0(y \wedge z \wedge u) - g(v, y)\phi_0(x \wedge z \wedge u) + g(v, z)\phi_0(x \wedge y \wedge u) \\ &\quad + g(z, u)\phi_0(v \wedge x \wedge y) - g(y, u)\phi_0(v \wedge x \wedge z) + g(x, u)\phi_0(v \wedge y \wedge z) \\ &\quad + g(u, x)\phi_0(y \wedge z \wedge v) - g(u, y)\phi_0(x \wedge z \wedge v) + g(u, z)\phi_0(x \wedge y \wedge v) \\ &\quad + g(z, v)\phi_0(u \wedge x \wedge y) - g(y, v)\phi_0(u \wedge x \wedge z) + g(x, v)\phi_0(u \wedge y \wedge z)\big] \\ &\quad = -12(u \vee v)(\varphi_0)(x \wedge y \wedge z) \end{aligned}$$

where $u \vee v$ acts on φ_0 via its Lie algebra action, being an element of the general linear Lie algebra of V. So

$$u \cdot \varphi_0 \vee v \cdot \varphi_0 \to -12(u \vee v)(\varphi_0)$$

which is a 3-form of type $S^2_0 V$.

4. $\Lambda^2 V \cdot \varphi_0 \otimes \varphi_0 \subset \Delta \otimes \Delta$ is generated by elements of the form $u \cdot \varphi_0 \wedge v \cdot \varphi_0$ which maps to a 3-form whose action on $x \wedge y \wedge z$ is

$$\frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(u)\varphi_0, \mathfrak{R}(v)\varphi_0) - \frac{1}{2} g(\mathfrak{R}(x \wedge y \wedge z)\mathfrak{R}(v)\varphi_0, \mathfrak{R}(u)\varphi_0) = 0$$

so $\Lambda^2 V \cdot \varphi_0 \otimes \varphi_0 = (\mathfrak{g}_2 \cdot \varphi_0 \otimes \varphi_0) \oplus (V \cdot \varphi_0 \otimes \varphi_0) \to 0$.

□

From Proposition 7.5.3 the kernel of the map $\Delta \otimes \Delta \to \Lambda^3 V$ can easily be deduced.

7.6 Covariant derivative formulae and type comparisons

Consider the covariant derivative $\nabla_U \varphi$ of the canonical spinor field φ of B_{G_2} with respect to a vector field U. Since φ has constant norm, $g(\nabla_U\varphi, \varphi) = 0$ for every U and so we can always write

$$\nabla_U \varphi = A_U \cdot \varphi$$

where A_U is a vector field depending linearly on U, i.e. A is a 1-form-valued 1-form on $\mathcal{M}$.
It will also be interesting to consider the cases when A is a 2-form-valued 1-form or a 3-form-valued 1-form (because of Lemma B.0.23, we will not consider higher-degree cases). In these cases A is a section of the bundle associated to $V \otimes \Lambda^2 V$ or $V \otimes \Lambda^3 V$, and using Proposition 7.3.4

$$\begin{aligned} V \otimes (\Lambda^2 V \cdot \varphi_0) &= V \otimes (\mathfrak{g}_2 \cdot \varphi_0) \oplus V \otimes (V \cdot \varphi_0) \\ &= V \otimes V \cdot \varphi_0 \\ V \otimes (\Lambda^3 V \cdot \varphi_0) &= V \otimes (\mathbb{R}\phi_0 \cdot \varphi_0) \oplus V \otimes (V \cdot \varphi_0) \oplus V \otimes (S^2_0 V \cdot \varphi_0) \\ &= (V \otimes \mathbb{R}\varphi_0) \oplus (V \otimes V \cdot \varphi_0) \ . \end{aligned}$$

However, since $g(\nabla\varphi, \varphi) = 0$, when A is a 3-form-valued 1-form it cannot have a piece corresponding to $V \otimes \mathbb{R}\varphi_0$. Thus we may always write

$$\nabla\varphi = A \cdot \varphi$$

where A is a 1-form with values in the 1-forms, 2-forms or 3-forms and where only the V-part of that 1-form, 2-form or 3-form matters (but we must remember that the first argument of A is *not* included in the Clifford multiplication here). Using the decomposition $V \otimes V = \mathbb{R}g \oplus V \oplus S^2_0 V \oplus \mathfrak{g}_2$ we will say

Definition 7.6.1. *The **type** of A is given by the summands of $V \otimes V$ present in $A \cdot \varphi$. When we write $\nabla\varphi = A \cdot \varphi$ it is obvious that the type of A is equal to the spin-class of B_{G_2}.*

We can say how the parts of A relate to the parts of $\nabla\varphi$:

$$\begin{aligned} A^{\mathbb{R}g} &= -\frac{1}{7} g(\mathcal{D}\varphi, \varphi) g \ , \\ A^V \cdot \varphi &= \big[-\frac{1}{7} g \cdot \mathcal{D}\varphi \big]_V + \big[\mathcal{T}\varphi \big]_V \ , \\ A^{S^2_0 V} \cdot \varphi &= \big[\mathcal{T}\varphi \big]_{S^2_0 V \cdot \varphi_0} \ , \\ A^{\mathfrak{g}_2} \cdot \varphi &= \big[\mathcal{T}\varphi \big]_{\mathfrak{g}_2 \cdot \varphi_0} \ . \end{aligned}$$

Consider the equation $\nabla_U\varphi = A_U \cdot \varphi$ where A_U is a 1-form, a 2-form or a 3-form. Can we derive a formula for $\nabla_U \phi$ in terms of A_U?

Theorem 7.6.2. *Let $\nabla_U\varphi = A_U \cdot \varphi$ be satisfied.*

1. *If A_U is a 1-form,*

$$\nabla_U \phi(X \wedge Y \wedge Z) = -8 * \phi(A_U \wedge X \wedge Y \wedge Z) \ ,$$

i.e.

$$\nabla_U \phi = -8 \iota_{A_U} * \phi \ .$$

2. *If A_U is a 2-form,*

$$\nabla_U \phi(X \wedge Y \wedge Z) = 4\phi\big(A_{U*}(X \wedge Y \wedge Z)\big) \ ,$$

i.e.

$$\nabla_U \phi = -4 A_{U*} \phi$$

where A_{U*} is the Lie algebra action of A_U (which is minus the pullback action ${A_U}^*$ since A_U is skew).

3. *If A_U is a 3-form,*

$$\nabla_U \phi(X \wedge Y \wedge Z) = -24 * \phi\big(A_{U*}(X \wedge Y \wedge Z)\big) + A_U(X \wedge Y \wedge Z) \ ,$$

i.e.

$$\nabla_U \phi = -24 {A_U}^* * \phi + A_U$$

where ${A_U}^*$ is the pullback of

$$A_{U*}(X \wedge Y \wedge Z) = \iota_X A_U \wedge Y \wedge Z + X \wedge \iota_Y A_U \wedge Z + X \wedge Y \wedge \iota_Z A_U \ .$$

Proof. We have

$$\begin{aligned}
\nabla_U \phi(X \wedge Y \wedge Z) &= \frac{1}{3!} \nabla_U g(\mathfrak{R}(\cdot)\varphi, \varphi)(X \wedge Y \wedge Z) \\
&= \frac{1}{3!} g(\mathfrak{R}(X \wedge Y \wedge Z)\nabla_U \varphi, \varphi) + \frac{1}{3!} g(\mathfrak{R}(X \wedge Y \wedge Z)\varphi, \nabla_U \varphi) \\
&= \frac{1}{3!} g(\mathfrak{R}(X \wedge Y \wedge Z)\mathfrak{R}(A_U)\varphi, \varphi) + \frac{1}{3!} g(\mathfrak{R}(X \wedge Y \wedge Z)\varphi, \mathfrak{R}(A_U)\varphi) \ .
\end{aligned}$$

Now if A_U is a 1-form or a 2-form, then when we move the $\mathfrak{R}(A_U)$ to the other side we pick up a minus sign. If A_U is a 3-form we get a plus sign. Thus

$$\nabla_U \phi(X \wedge Y \wedge Z) = -\frac{1}{3!} g([\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z)]\varphi, \varphi)$$

for A_U a 1-form or a 2-form and where $[\cdot, \cdot]$ is the commutator. Also

$$\nabla_U \phi(X \wedge Y \wedge Z) = \frac{1}{3!} g(\{\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z)\}\varphi, \varphi)$$

when A_U is a 3-form and where $\{\cdot, \cdot\}$ is the anticommutator. Lemma B.0.22 implies

$$[\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z)] = 2\mathfrak{R}(A_U \wedge X \wedge Y \wedge Z)$$

for a 1-form,

$$[\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z)] = -4\mathfrak{R}\big(A_{U*}(X \wedge Y \wedge Z)\big)$$

for a 2-form (where A_U acts by its Lie algebra action), and

$$\{\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z)\} = -6\mathfrak{R}\big(A_{U*}(X \wedge Y \wedge Z) - g(A_U, X \wedge Y \wedge Z)\big)$$

for a 3-form, where

$$A_{U*}(X \wedge Y \wedge Z) = \iota_X A_U \wedge Y \wedge Z + X \wedge \iota_Y A_U \wedge Z + X \wedge Y \wedge \iota_Z A_U \ .$$

Apply Proposition 7.5.2 to get the formulae. □

Theorem 7.6.3. *The form-class of a G_2-structure is equal to its spin-class.*

Proof. Suppose the spin-class is $\mathbb{R}$, so $A_U = \lambda U$ for $\lambda \in \mathbb{R}$. Then Theorem 7.6.2 part *1* tells us

$$\nabla_U \phi = -8\lambda \iota_U * \phi$$

i.e.

$$\nabla \phi = -8\lambda * \phi$$

so the form-class is $\mathbb{R}$. Now suppose the spin-class is $S^2_0 V$. To show the $\mathbb{R}$, $\mathfrak{g}_2$ and V-parts of $\nabla\phi$ vanish, we have to prove $g(d\phi, *\phi) = 0$ and $\delta\phi = 0$.

$$\begin{aligned} 4d\phi(X \wedge Y \wedge Z \wedge W) &= \nabla_X \phi(Y \wedge Z \wedge W) + \nabla_Y \phi(Z \wedge W \wedge X) \\ &\quad + \nabla_Z \phi(W \wedge X \wedge Y) + \nabla_W \phi(X \wedge Y \wedge Z) \\ &= -8\big[* \phi(A_X \wedge Y \wedge Z \wedge W) + *\phi(A_Y \wedge Z \wedge W \wedge X) \\ &\quad + *\phi(A_Z \wedge W \wedge X \wedge Y) + *\phi(A_W \wedge X \wedge Y \wedge Z)\big] \ , \end{aligned}$$

and so

$$\begin{aligned} g(d\phi, *\phi) &= \sum_{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}=1}^{7} d\phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) * \phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \\ &= -2 \sum_{\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}=1}^{7} *\phi(A_\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) * \phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \\ &\qquad + *\phi(A_\mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l} \wedge \mathfrak{i}) * \phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \\ &\qquad + *\phi(A_\mathfrak{k} \wedge \mathfrak{l} \wedge \mathfrak{i} \wedge \mathfrak{j}) * \phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \\ &\qquad + *\phi(A_\mathfrak{l} \wedge \mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k}) * \phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \ . \end{aligned}$$

Since A has type $S^2_0 V$, $A_\mathfrak{i}$ is always orthogonal to $\mathfrak{i}$ and since no two terms in the expression for $*\phi$ contain three elements the same from the basis $\mathfrak{1}, \ldots, \mathfrak{7}$, we see that

$$*\phi(\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) \neq 0 \ \Rightarrow \ *\phi(A_\mathfrak{i} \wedge \mathfrak{j} \wedge \mathfrak{k} \wedge \mathfrak{l}) = 0 \ .$$

Thus $g(d\phi, *\phi) = 0$. Furthermore

$$\delta\phi = \sum_{\mathfrak{i}=1}^{7} \iota_{\mathfrak{i}} \nabla_{\mathfrak{i}} \phi = -8 \sum_{\mathfrak{i}=1}^{7} *\phi(A_{\mathfrak{i}} \wedge \mathfrak{i} \wedge \cdot) \ .$$

Since A has type $S_0^2 V$, and

$$\begin{aligned}\sum_{\mathfrak{i}=1}^{7} &*\phi(\mathfrak{j} \vee \mathfrak{k}(\mathfrak{i}) \wedge \mathfrak{i} \wedge \cdot) \\ &= \frac{1}{2} * \phi(\mathfrak{k} \wedge \mathfrak{j} \wedge \cdot) + \frac{1}{2} * \phi(\mathfrak{j} \wedge \mathfrak{k} \wedge \cdot) \\ &= 0 \ ,\end{aligned}$$

we find $\delta\phi = 0$. The form-class is therefore $S_0^2 V$. Now suppose the spin-class is $\mathfrak{g}_2$. Since A is skew,

$$d\phi = -2A(*\phi)$$

where A acts by its Lie algebra action. When A has type $\mathfrak{g}_2$, $d\phi = 0$ so the form-class is $\mathfrak{g}_2$. In fact, we can show that

$$\delta\phi = \frac{2}{3} A \ ,$$

in a similar way to the V case below. Now suppose the spin-class is V. We have to prove that the $\mathbb{R}$ and $S_0^2 V$ components of $d\phi$ and the $\mathfrak{g}_2$-part of $\delta\phi$ vanish. Since $A_{\mathfrak{i}}$ is always orthogonal to $\mathfrak{i}$, the same argument as in the $S_0^2 V$ case works to show $g(d\phi, *\phi) = 0$. Using the basis

$$\begin{gathered} 1 \wedge 2 + 4 \wedge 7 - 5 \wedge 6 \ , \ 1 \wedge 3 - 4 \wedge 6 - 5 \wedge 7 \ , \ 1 \wedge 4 - 2 \wedge 7 + 3 \wedge 6 \ , \\ 1 \wedge 5 + 2 \wedge 6 + 3 \wedge 7 \ , \ 1 \wedge 6 - 2 \wedge 5 - 3 \wedge 4 \ , \ 1 \wedge 7 + 2 \wedge 4 - 3 \wedge 5 \ , \\ 2 \wedge 3 + 4 \wedge 5 - 6 \wedge 7 \end{gathered}$$

for the subspace $V \subset \Lambda^2 V$, it is easy to check that

$$\sum_{\mathfrak{i}=1}^{7} *\phi(A_{\mathfrak{i}} \wedge \mathfrak{i} \wedge \cdot) = \frac{1}{6} A$$

when A has type V. Thus

$$\delta\phi = -\frac{4}{3} A$$

and the $\mathfrak{g}_2$-part vanishes. We can use the same basis to prove $A(*\phi)$ has type V when A does:

$$(1 \wedge 2 + 4 \wedge 7 - 5 \wedge 6)(*\phi) = -\frac{3}{2} 3 \wedge \phi \ .$$

Thus, using the fact that when A is skew we have $d\phi = -2A(*\phi)$, we deduce that when A has type V the $S_0^2 V$-part of $d\phi$ vanishes and the form-class is V. Now that we have shown that the spin-class and form-class of a G_2-structure agree for types $\mathbb{R}$, $S_0^2 V$, $\mathfrak{g}_2$ and V, we remark that the statement for the other classes follows by linearity. $\square$

Chapter 8

$Spin_7^+$-structures

The second of the two exceptional restricted holonomy groups of locally irreducible non-locally symmetric spaces appearing on Berger's list is the Lie group $Spin_7^+$. Local existence of spaces with $Spin_7^+$ holonomy was established in the same papers [Bry87] and [BS89] as for the G_2 case, and compact examples in [Joy96a]. This chapter will contain the analogous theory for $Spin_7^+$ as the previous one did for G_2. However, it will be better to introduce the group $Spin_7^+$ in a different way.

8.1 The group $Spin_7^+$ and its representations

The easiest way to deal with $Spin_7^+$ is by introducing three of its representations at once. We know that the real Clifford algebra $C\ell_7$ is non-canonically isomorphic to the sum of real matrix algebras $\mathbb{R}(8) \oplus \mathbb{R}(8)$, and the group $Spin_8$ is contained in $C\ell_7$. From this it is clear that the two real irreducible spin representations of $Spin_8$ are eight dimensional. We denote them by $\mathfrak{L}$ and $\mathfrak{R}$, both acting on $\Delta \stackrel{\text{def}}{=} \mathbb{O}$ as explained in the G_2 chapter, except we shall write Δ^+ when using $\mathfrak{R}$ and Δ^- when using $\mathfrak{L}$. The standard double covering $\eta : Spin_8 \to SO_8$ gives us a third inequivalent irreducible real representation of dimension eight called (η, W) (W is just $\mathbb{O}$ as well, but we use a different symbol so we remember to think of its elements as 'vectors' and not spinors).

As is explained in [LM89], there is an outer automorphism σ of $Spin_8$ such that σ^3 is the identity. This map σ is called the *triality automorphism*, and it swaps around the three representations explained above as follows:

$$\eta \circ \sigma = \mathfrak{L}\ ,\ \ \mathfrak{L} \circ \sigma = \mathfrak{R}\ ,\ \ \mathfrak{R} \circ \sigma = \eta\ .$$

Now let $\iota : Spin_7 \to Spin_8$ be the standard inclusion, i.e. the one so that

$$\begin{tikzcd} Spin_7 \arrow[r, "\iota"] \arrow[d, "\eta'"] & Spin_8 \arrow[d, "\eta"] \\ SO_7 \arrow[r, "\iota'"] & SO_8 \end{tikzcd}$$

commutes, where $\eta' : Spin_7 \to SO_7$ is the standard double cover and $\iota' : SO_7 \to SO_8$ is the standard inclusion [1]. Then

Lemma 8.1.1. *When η, $\mathfrak{L}$ and $\mathfrak{R}$ are restricted to $\iota Spin_7 \subset Spin_8$,*

1. *(η, W) splits into irreducibles of dimensions one and seven,*

2. *$\mathfrak{L}$ and $\mathfrak{R}$ both remain irreducible and become equivalent.*

Proof. 1. Remember W is just $\mathbb{O}$, and $\eta \circ \iota : Spin_7 \to SO_7$ leaves the 'first' dimension $\mathrm{Re}\mathbb{O}$ invariant pointwise,

2. See [LM89].

$\square$

Definition 8.1.2. *In line with our notation for G_2, use the notation $\widetilde{Spin_7^+}$ for the image of $\iota Spin_7 \subset Spin_8$ under the triality automorphism σ:*

$$\widetilde{Spin_7^+} \stackrel{\text{def}}{=} \sigma \circ \iota Spin_7 \subset Spin_8$$

(so the tilde indicates that the space lives in the spin group instead of the special orthogonal group). The kernel of $\eta : Spin_8 \to SO_8$ trivially intersects $\sigma \circ \iota Spin_7$ (σ moves -1) and so η is injective on $\widetilde{Spin_7^+}$; we define

$$Spin_7^+ \stackrel{\text{def}}{=} \eta(\widetilde{Spin_7^+}) \subset SO_8 \ .$$

Note that $\iota Spin_7$, $\widetilde{Spin_7^+}$ and $Spin_7^+$ are all isomorphic to $Spin_7$, but sit in their surroundings in different ways.

Theorem 8.1.3. *The restriction of the real spin representation $(\mathfrak{R}, \Delta^+)$ to $\widetilde{Spin_7^+} \subset Spin_8$ has an invariant element Ψ_0, which we shall call the **canonical spinor**.*

Proof. Note $\mathfrak{R}|_{\widetilde{Spin_7^+}} = \mathfrak{R}|_{\sigma\circ\iota Spin_7} = \mathfrak{R} \circ \sigma|_{\iota Spin_7} = \eta|_{\iota Spin_7}$ and apply Lemma 8.1.1. $\square$

We should also note that upon restriction to $\widetilde{Spin_7^+} \subset Spin_8$, W and Δ^- become equivalent. We will write

$$\Delta^+ = \mathbb{R}\Psi_0 \oplus \Psi_0^\perp$$

where $\Psi_0^\perp$ is the irreducible 7-dimensional representation of $Spin_7^+$.

For the vector space V in the case of G_2 we used the basis $\mathfrak{1}, \ldots, \mathfrak{7}$, because V is the space of imaginary octonions. For W we shall use the basis $\mathfrak{0}, \mathfrak{1}, \ldots, \mathfrak{7}$ where $\mathfrak{0}$ is our new name for the element $1 \in \mathbb{O}$.

The real Clifford algebra in dimension eight

$$C\ell_8 \stackrel{\text{non-canon}}{\cong} \mathbb{R}(16)$$

has even subalgebra

$$C\ell_8^0 \stackrel{\text{non-canon}}{\cong} \mathbb{R}(8) \oplus \mathbb{R}(8)$$

and there is an isomorphism

$$C\ell_7 \xrightarrow{\sim} C\ell_8^0 = C\ell_7^0 + \mathfrak{0}C\ell_7^1$$

[1] Actually for us, $\mathbb{R}^7$ is included in $\mathbb{R}^8$ as the *final* seven dimensions so that we can write the splitting $\mathbb{O} = \mathrm{Re}\mathbb{O} \oplus \mathrm{Im}\mathbb{O}$ in the right order.

where $\mathfrak{o}$ is our new vector. To define Δ^+ and Δ^- as representations of $Spin_8$, we first included $Spin_8$ inside $C\ell_7$ using the above isomorphism (though we did not say so). The full real Clifford representation of $C\ell_8$ is $\Delta^+ + \Delta^-$ and we can now explain how Clifford multiplication works. Since Δ^+ and Δ^- are naturally isomorphic as vector spaces (they are $\mathbb{O}$ with different actions $\mathfrak{L}$ and $\mathfrak{R}$) we can use Lemma 4.5.1 to deduce

$$v \cdot x = \mathfrak{R}(v)x$$

for all $v \in V \subset W$, $x \in \Delta^+ = \mathbb{O}$ and where we consider the result $\mathfrak{R}(v)x$ to be an element of Δ^- using the natural identification. Also

$$\mathfrak{o} \cdot x = -x \ , \ \mathfrak{o} \cdot y = y$$

for $x \in \Delta^+$, $y \in \Delta^-$, although again we consider the resulting x and y to be in the opposite spaces by the natural identification. We will need these facts to prove some properties of Clifford multiplication and the squaring map later on.

Definition 8.1.4. *The **canonical form** $\Phi_0 : \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \to \mathbb{R}$ is defined by*

$$\begin{aligned} \Phi_0(\mathfrak{o}, y, z, w) &\stackrel{\text{def}}{=} -\frac{1}{24}\mathrm{Re}\,\underrightarrow{yzw} \quad , \\ \Phi_0(x, y, z, w) &\stackrel{\text{def}}{=} \ \ \frac{1}{24}\mathrm{Re}\,\underrightarrow{xyzw} \end{aligned}$$

where x, y, z, w lie in the subspace $\mathrm{Im}\mathbb{O}$. The factor of $1/24$ and the sign will be convenient later.

Proposition 8.1.5. *We have*

1. *Φ_0 is totally skew,*
2. *The stabiliser of Φ_0 is $Spin_7^+$.*

Proof.
1. Similar to Proposition 7.1.2, using Lemma A.0.14. So Φ_0 may be considered as a map $\Lambda^4 W \to \mathbb{R}$.
2. See [Sal89].

$\square$

We can find the action of Φ_0 on simple elements and show that

$$\begin{aligned} \Phi_0 = {} & \mathfrak{o} \wedge 1 \wedge 2 \wedge 3 + \mathfrak{o} \wedge 1 \wedge 4 \wedge 5 - \mathfrak{o} \wedge 1 \wedge 6 \wedge 7 + \mathfrak{o} \wedge 2 \wedge 4 \wedge 6 \\ & + \mathfrak{o} \wedge 2 \wedge 5 \wedge 7 + \mathfrak{o} \wedge 3 \wedge 4 \wedge 7 - \mathfrak{o} \wedge 3 \wedge 5 \wedge 6 \\ & + 4 \wedge 5 \wedge 6 \wedge 7 + 2 \wedge 3 \wedge 6 \wedge 7 - 2 \wedge 3 \wedge 4 \wedge 5 + 1 \wedge 3 \wedge 5 \wedge 7 \\ & + 1 \wedge 3 \wedge 4 \wedge 6 + 1 \wedge 2 \wedge 5 \wedge 6 - 1 \wedge 2 \wedge 4 \wedge 7 \ . \end{aligned}$$

Proposition 8.1.6. *The canonical form Φ_0 satisfies the following:*

1. *$\Phi_0 = \mathfrak{o} \wedge \phi_0 + *_7\phi_0$*
2. *$*_8\Phi_0 = \Phi_0$*

where ϕ_0 is the canonical 3-form in dimension seven.

Proof. The first point is clear from the above expression for Φ_0 in terms of the basis $0, 1, \ldots, 7$. For the second part, Proposition B.0.24 gives

$$*_8\Phi_0 = *_8(0 \wedge \phi_0) + *_8 *_7 \phi_0 = *_7\phi_0 + 0 \wedge *_7 *_7 \phi_0 = *_7\phi_0 + 0 \wedge \phi_0 = \Phi_0 \ .$$

□

Remark 8.1.7. *Proposition 8.1.6, along with Proposition B.0.25, is used by Cabrera [Cab95b] to define $Spin_7^+$-structures on principal bundles with fibre-dimension 1 over manifolds with G_2-structure. Cabrera then relates the types of the structures. We can do the same with a warped product $\mathcal{M} = \mathbb{R} \times_{f^2} \mathcal{B}$ where $\mathcal{B}$ possesses a G_2-structure.*

Denote by $\mathfrak{spin}_7^+$ the Lie algebra of $Spin_7^+$, so $\mathfrak{spin}_7^+ \subset \mathfrak{so}_8$. A basis for $\mathfrak{so}_8$ is $0\wedge 1, 0\wedge 2, \ldots, 6\wedge 7$ and just as in the G_2 case we get the Lie algebra actions

$$\begin{aligned}
0 \wedge 1(\Phi_0) &= -2 \wedge 3(\Phi_0) = -4 \wedge 5(\Phi_0) = 6 \wedge 7(\Phi_0) \ ,\\
0 \wedge 2(\Phi_0) &= 1 \wedge 3(\Phi_0) = -4 \wedge 6(\Phi_0) = -5 \wedge 7(\Phi_0) \ ,\\
0 \wedge 3(\Phi_0) &= -1 \wedge 2(\Phi_0) = -4 \wedge 7(\Phi_0) = 5 \wedge 6(\Phi_0) \ ,\\
0 \wedge 4(\Phi_0) &= 1 \wedge 5(\Phi_0) = 2 \wedge 6(\Phi_0) = 3 \wedge 7(\Phi_0) \ ,\\
0 \wedge 5(\Phi_0) &= -1 \wedge 4(\Phi_0) = 2 \wedge 7(\Phi_0) = -3 \wedge 6(\Phi_0) \ ,\\
0 \wedge 6(\Phi_0) &= -1 \wedge 7(\Phi_0) = -2 \wedge 4(\Phi_0) = 3 \wedge 5(\Phi_0) \ ,\\
0 \wedge 7(\Phi_0) &= 1 \wedge 6(\Phi_0) = -2 \wedge 5(\Phi_0) = -3 \wedge 4(\Phi_0) \ .
\end{aligned}$$

It is clear that $\mathfrak{spin}_7^+$ can be identified as the subalgebra of $\mathfrak{so}_8$ which sends Φ_0 to zero; a basis of $\mathfrak{spin}_7^+$ can be read off as

$$\begin{gathered}
0 \wedge 1 + 2 \wedge 3 \ , \ 0 \wedge 1 + 4 \wedge 5 \ , \ 0 \wedge 1 - 6 \wedge 7 \ ,\\
0 \wedge 2 - 1 \wedge 3 \ , \ 0 \wedge 2 + 4 \wedge 6 \ , \ 0 \wedge 2 + 5 \wedge 7 \ ,\\
0 \wedge 3 + 1 \wedge 2 \ , \ 0 \wedge 3 + 4 \wedge 7 \ , \ 0 \wedge 3 - 5 \wedge 6 \ ,\\
0 \wedge 4 - 1 \wedge 5 \ , \ 0 \wedge 4 - 2 \wedge 6 \ , \ 0 \wedge 4 - 3 \wedge 7 \ ,\\
0 \wedge 5 + 1 \wedge 4 \ , \ 0 \wedge 5 - 2 \wedge 7 \ , \ 0 \wedge 5 + 3 \wedge 6 \ ,\\
0 \wedge 6 + 1 \wedge 7 \ , \ 0 \wedge 6 + 2 \wedge 4 \ , \ 0 \wedge 6 - 3 \wedge 5 \ ,\\
0 \wedge 7 - 1 \wedge 6 \ , \ 0 \wedge 7 + 2 \wedge 5 \ , \ 0 \wedge 7 + 3 \wedge 4 \ .
\end{gathered}$$

Notice how similar this is to the expression for Φ_0. We can see $\dim_{\mathbb{R}} \mathfrak{spin}_7^+ = 21$ and so $Spin_7^+$ is also 21-dimensional.

Definition 8.1.8. *A* ***Cayley frame*** *of W is a frame obtained from $0, \ldots, 7$ by acting with an element of $Spin_7^+$.*

The Cayley frames form a $Spin_7^+$-subtorsor of the torsor of frames of W. They are very special orthonormal frames—in an arbitrary orthonormal frame the above calculations may become very complicated.

Proposition 8.1.9. *Upon restriction to $Spin_7^+ \subset Spin_8$ we have the following decompositions into irreducibles:*

1.
$$\Lambda^2 W = \mathfrak{spin}_7^+ \oplus \Psi_0^\perp$$

where $\Psi_0^\perp$ *is the seven dimensional irreducible real representation of* $Spin_7^+$*.*

2.
$$\Lambda^3 W = W \oplus F$$

where

$$W \cong \left\{ \iota_w \Phi_0 \in \Lambda^3 W \mid w \in W \right\}$$
$$F \cong \left\{ \beta \in \Lambda^3 W \mid \beta \wedge \Phi_0 = 0 \right\} .$$

and F *is* 48*-dimensional.*

3.
$$\Lambda^4 W = \mathbb{R}\Phi_0 \oplus \Psi_0^\perp \oplus S_0^2 \Psi_0^\perp \oplus \Lambda_-^4 W$$

where in this case

$$\Psi_0^\perp \cong \left\{ (u \wedge v)(\Phi_0) \in \Lambda^4 W \mid u \wedge v \in \Psi_0^\perp \subset \Lambda^2 W \right\}$$

and $(u \wedge v)(\Phi_0)$ *is the Lie algebra action of* $u \wedge v$ *on* Φ_0*, and* $\Lambda_-^4 W$ *is the* -1*-eigenspace of the Hodge star operator* $*_8$ *(the other summands make up* $\Lambda_+^4 W$*).*

Proof. These are proved in [FG82], [Fer86]. □

From the basis for $\mathfrak{spin}_7^+$ above, we can easily produce a basis for the complement $\Psi_0^\perp \subset \Lambda^2 W$:

$$\begin{aligned}
&0 \wedge 1 - 2 \wedge 3 - 4 \wedge 5 + 6 \wedge 7 \ , \ 0 \wedge 2 + 1 \wedge 3 - 4 \wedge 6 - 5 \wedge 7 \ , \\
&0 \wedge 3 - 1 \wedge 2 - 4 \wedge 7 + 5 \wedge 6 \ , \ 0 \wedge 4 + 1 \wedge 5 + 2 \wedge 6 + 3 \wedge 7 \ , \\
&0 \wedge 5 - 1 \wedge 4 + 2 \wedge 7 - 3 \wedge 6 \ , \ 0 \wedge 6 - 1 \wedge 7 - 2 \wedge 4 + 3 \wedge 5 \ , \\
&0 \wedge 7 + 1 \wedge 6 - 2 \wedge 5 - 3 \wedge 4 \ .
\end{aligned}$$

Proposition 8.1.10. *We have*

1. *The action of vectors on* Ψ_0 *by Clifford multiplication yields an isomorphism of* $Spin_7^+$*-representations* $W \xrightarrow{\sim} \Delta^-$*.*

2. *The action of* 2*-forms on* Ψ_0 *by Clifford multiplication yields a homomorphism* $\Lambda^2 W \to \Psi_0^\perp \subset \Delta^+$ *such that*
$$\mathfrak{spin}_7^+ \to 0 \ , \ \Psi_0^\perp \xrightarrow{\sim} \Psi_0^\perp$$
where we've used the simple notation $\Lambda^2 W = \mathfrak{spin}_7^+ \oplus \Psi_0^\perp$ *for the* $Spin_7^+$*-decomposition of* 2*-forms.*

3. *The action of* 3*-forms on* Ψ_0 *by Clifford multiplication yields a homomorphism* $\Lambda^3 W \to \Delta^-$ *such that*
$$W \xrightarrow{\sim} \Delta^- \ , \ F \to 0$$
where we've used the simple notation $\Lambda^3 W = W \oplus F$ *for the* $Spin_7^+$*-decomposition of* 3*-forms.*

4. *The action of 4-forms on Ψ_0 by Clifford multiplication yields a homomorphism $\Lambda^4 W \to \Delta^+$ such that*

$$\mathbb{R}\Phi_0 \xrightarrow{\sim} \mathbb{R}\Psi_0 \ , \ \Psi_0^\perp \xrightarrow{\sim} \Psi_0^\perp \ , \ S_0^2\Psi_0^\perp \to 0 \ , \ \Lambda_-^4 W \to 0$$

where we've used the simple notation $\Lambda^4 W = \mathbb{R}\Phi_0 \oplus \Psi_0^\perp \oplus S_0^2\Psi_0^\perp \oplus \Lambda_-^4 W$ for the $Spin_7^+$-decomposition of 4-forms.

Proof. 1. W acts by isomorphisms since g is definite. Apply Schur's Lemma.

2. Obvious using the identification of $\mathfrak{spin}_8$ with $\Lambda^2 W$.

3. It is obvious that F maps to 0 by Schur's Lemma. To show $W \xrightarrow{\sim} \Delta^-$ we have to find a 3-form of type W which doesn't map to zero, and then the result follows also by Schur's Lemma. Such a 3-form is $4\iota_{\mathtt{o}}\Phi_0 = \phi_0$, and we can easily check $\phi_0 \cdot \Psi_0$ is a non-zero element of Δ^-.

4. $S_0^2\Psi_0^\perp \to 0$ and $\Lambda_-^4 W \to 0$ must hold by Schur's Lemma. It is easy to show that $\Phi_0 \cdot \Psi_0 = 14\Psi_0$. Finally, a 4-form of type $\Psi_0^\perp$ is of the form $(u \wedge v)(\Phi_0)$ where $u \wedge v$ is in the subspace of $\Lambda^2 W$ of type $\Psi_0^\perp$ (the complement of $\mathfrak{spin}_7^+ \subset \mathfrak{spin}_8$). Now

$$(\mathtt{o} \wedge \mathtt{1})(\Phi_0) \cdot \Psi_0 = 8\mathtt{o}\mathtt{1} \cdot \Psi_0$$

is a non-zero element of $\Psi_0^\perp$, because $\mathtt{o}$ is the natural identification of vector spaces $\Delta^- \to \Delta^+$ as explained above.

$\square$

8.2 $Spin_7^+$-structures

Definition 8.2.1. *A $Spin_7^+$-**structure** $B_{Spin_7^+}$ on $\mathcal{M}^8$ is a $Spin_7^+$-subbundle of the bundle of frames $GL(\mathcal{M})$ of $\mathcal{M}$. As for G_2-structures, we call the elements (and local sections) of $B_{Spin_7^+}$ the **Cayley frames** of $\mathcal{M}$.*

A $Spin_7^+$-structure $B_{Spin_7^+}$ also brings with it a rich geometry. Since $Spin_7^+$ lies inside SO_8, the natural enlargement

$$B_{Spin_7^+} \times_{Spin_7^+} SO_8 \subset GL(\mathcal{M})$$

is a Riemannian structure, i.e. it defines a Riemannian metric g on $\mathcal{M}$. Furthermore[2]

Theorem 8.2.2. *([LM89]) A smooth manifold $\mathcal{M}^8$ admits a $Spin_7^+$-structure if and only if it is spin* and *either of the Euler classes $\chi(S\mathcal{M}^+)$ or $\chi(S\mathcal{M}^-)$ vanishes. Equivalently, $\mathcal{M}^8$ admits a $Spin_7^+$-structure if and only if it is spin* and *for one of the two choices of orientation,*

$$p_1(\mathcal{M})^2 - 4p_2(\mathcal{M}) + 8\chi(\mathcal{M}) = 0$$

where $p_i(\mathcal{M})$ is the ith Pontryjagin class of $\mathcal{M}$.

Again we can say slightly more:

$$B_{Spin_7^+} \times_{\widetilde{Spin_7^+}} Spin_8$$

is a spin structure on $\mathcal{M}$.

[2]This obstruction was originally proved in the context of cross products by Gray [Gra69].

We've explained that for (η, W) the 8-dimensional irreducible real representation of $Spin_7^+ < SO_8$, there is an invariant element $\Phi_0 \in \Lambda^4 W$. Since $\Lambda^4 T\mathcal{M} \cong B_{Spin_7^+} \times_{Spin_7^+} \Lambda^4 W$ is an isomorphism of associated bundles we get a canonical 4-form Φ on $\mathcal{M}$. By *2* of Lemma 8.1.5, the stabliser of Φ_0 is $Spin_7^+$ and we can recover $B_{Spin_7^+}$ from Φ. In a Cayley frame $0, 1, \ldots, 7$,

$$\begin{aligned}
\Phi = {} & 0 \wedge 1 \wedge 2 \wedge 3 + 0 \wedge 1 \wedge 4 \wedge 5 - 0 \wedge 1 \wedge 6 \wedge 7 + 0 \wedge 2 \wedge 4 \wedge 6 \\
& + 0 \wedge 2 \wedge 5 \wedge 7 + 0 \wedge 3 \wedge 4 \wedge 7 - 0 \wedge 3 \wedge 5 \wedge 6 \\
& + 4 \wedge 5 \wedge 6 \wedge 7 + 2 \wedge 3 \wedge 6 \wedge 7 - 2 \wedge 3 \wedge 4 \wedge 5 + 1 \wedge 3 \wedge 5 \wedge 7 \\
& + 1 \wedge 3 \wedge 4 \wedge 6 + 1 \wedge 2 \wedge 5 \wedge 6 - 1 \wedge 2 \wedge 4 \wedge 7 \ .
\end{aligned}$$

Denote by ∇ the Levi-Civita covariant derivative associated to the metric g on $\mathcal{M}$. In the paper of Fernández [Fer86], $Spin_7^+$-structures are divided into 4 types based on splitting $\nabla\Phi$ into components. We outline this procedure here.

The derivative $\nabla\Phi$ is a section of the associated bundle

$$T\mathcal{M} \otimes \Lambda^4 T\mathcal{M} = B_{Spin_7^+} \times_{Spin_7^+} (W \otimes \Lambda^4 W)$$

which splits into irreducible subbundles corresponding to the irreducible summands of the $Spin_7^+$-representation $W \otimes \Lambda^4 W$. As an SO_8-representation

$$W \otimes \Lambda^4 W = \Lambda^5 W \oplus \Lambda^3 W \oplus \Lambda^{4,1} W$$

where the final summand is defined merely to be the orthogonal complement of the first two. To prove this we must write down all the necessary equivariant maps which split the exact sequences resulting from the projections onto the different summands. We have done this and omit the proofs. In the bundle picture, we can use this decomposition on $\nabla\Phi$ and this gives us

$$\nabla\Phi = d\Phi + \frac{4}{5} g \wedge \delta\Phi + \mathcal{T}\Phi \ .$$

The Hodge star $*_8$ gives us an isomorphism $\Lambda^5 W \cong \Lambda^3 W$, and we also have the splitting given in Proposition 8.1.9

$$\Lambda^3 W = W \oplus F$$

and $\Lambda^{4,1} W$ splits as

$$\Lambda^{4,1} W = W \oplus F \oplus C_1^{112} \oplus C_2^{112} \oplus C^{168}$$

where C_1^{112}, C_2^{112} and C^{168} are irreducibles of the denoted dimensions. Thus, as a representation of $Spin_7^+$,

$$W \otimes \Lambda^4 W = (W \oplus F) \oplus (W \oplus F) \oplus (W \oplus F \oplus C_1^{112} \oplus C_2^{112} \oplus C^{168}) \ .$$

The resulting decomposition of a single element of $W \otimes \Lambda^4 W$ is very complicated to write down, and so we omit it. However, when we apply this decomposition to $\nabla\Phi$ we find

$$\begin{aligned}
\nabla\Phi = {} & -\frac{24}{7} d\Phi(\Phi) \wedge \Phi + \left[*_8 \, d\Phi \right]_F \\
& - \frac{384}{35} g \wedge \Phi(\Phi(\delta\Phi)) + \frac{4}{5} g \wedge \left[\delta\Phi \right]_F + \mathcal{T}\Phi
\end{aligned}$$

where $\mathcal{T}\Phi$ is the $\Lambda^{4,1}W$ part all in one piece.

Lemma 8.2.3. *The parts $d\Phi$ and $\delta\Phi$ are proportional.*

Proof. It follows from $*_8\Phi = \Phi$ that

$$\delta\Phi = -\frac{1}{4} *_8 d\Phi \ .$$

We can also equate the W-parts: $\Phi(\delta\Phi) = -\frac{1}{4}d\Phi(\Phi)$. □

Theorem D.0.39 tells us that $\nabla\Phi$ is identified with the torsion of the $Spin_7^+$-structure, and this takes values in the bundle associated to $\mathfrak{spin}_7^{+\perp} \otimes W$, where $\mathfrak{spin}_7^{+\perp}$ is the orthogonal complement of $\mathfrak{spin}_7^+ \subset \mathfrak{so}_8$. Proposition 8.1.9 shows that $\mathfrak{spin}_7^{+\perp} = \Psi_0^\perp$. According to [Sal89],

$$W \otimes W = W \oplus W \oplus F$$

and this means the torsion may only have W and F parts. The components of $\nabla\Phi$ corresponding to the irreducibles C_1^{112}, C_2^{112} and C^{168} must vanish. David [Dav11] has shown that the W and F parts appearing in $\mathcal{T}\Phi$ are proportional to those already appearing in the other terms, and therefore

Theorem 8.2.4. *([Dav11]) The canonical 4-form Φ is conformal-Killing if and only if it is parallel.*

So there are two components of $\nabla\Phi$ which are important. We can classify $Spin_7^+$-structures by which of those components are zero or not, and this gives us four *form-classes*. Each form-class can be described with a differential equation that the canonical form must satisfy, and we tabulate these below. This was first achieved by Fernández [Fer86].

Theorem 8.2.5. *([Fer86]) The **form-classes** of $Spin_7^+$-structures are given by the following relations. The notation of the following table matches ours, and so differs considerably from the paper [Fer86].*

Form-class	*Defining relations*
0	$\nabla\Phi = 0$
W	$\nabla\Phi = -\frac{24}{7}d\Phi(\Phi) \wedge \Phi - \frac{384}{35}g \wedge \Phi(\Phi(\delta\Phi))$
F	$d\Phi(\Phi) = 0 = \Phi(\delta\Phi)$
$W \oplus F$	*No relation*

Proof. Clear from decomposition. □

Theorem 8.2.6. *([Fer86, Fer87, Cab95a, Joy96a]) All the form-classes of $Spin_7^+$-structures occur, and moreover they occur with compact examples.*

A natural question to ask is, which of the classes are conformally invariant? This was answered by Fernández [Fer86] and the answer is that a class is conformally invariant if it contains a W summand. The $Spin_7^+$-structures of type W are known as *locally conformally-parallel* because a particular local conformal change of metric (resulting from defining a new 4-form as a positive function times the old one) makes such a structure parallel.

Many of the other classes have analogues in the classification of almost-Hermitian structures given by Gray and Hervella [GH80]. The similarity is no surprise given the unifying perspective explained in Appendix C. It may be possible to formulate some deep questions about $Spin_7^+$-structures, such as an analogue of the Calabi conjecture, but this has not yet been achieved.

Note that a manifold with $Spin_7^+$-structure has a naturally-defined vector field, given by its W-part. Specifically,

Definition 8.2.7. *The **canonical vector field** θ on a manifold $\mathcal{M}$ with $Spin_7^+$-structure is the projection of $*d\Phi$ onto $T\mathcal{M}$:*

$$\theta \stackrel{\text{def}}{=} -210 d\Phi(\Phi) \ .$$

Using Lemma 8.2.3 we see that the projection of $\delta\Phi$ onto $T\mathcal{M}$ is $-\frac{1}{4}\theta$.

Lemma 8.2.8. *The W-part of $d\Phi$ is*

$$\frac{1}{7}\theta \wedge \Phi \ .$$

Proof. This follows easily from Lemma B.0.17. □

Now we have classified $Spin_7^+$-structures in terms of their canonical 4-form Φ, we want to do the same in terms of the canonical spinor field Ψ and then show that the two classifications agree using the squaring map.

8.3 The spinorial classification

Using the $Spin_7^+$-invariant element $\Psi_0 \in \Delta^+$ we get a natural spinor field on $\mathcal{M}$, denoted Ψ. The real spinor bundle of $\mathcal{M}$ (corresponding to the spin structure specified by $B_{Spin_7^+}$) breaks into irreducibles

$$\begin{aligned} S\mathcal{M} &\stackrel{\text{def}}{=} B_{Spin_7^+} \times_{\widetilde{Spin_7^+}} \Delta^+ \oplus \Delta^- \\ &= B_{Spin_7^+} \times_{\widetilde{Spin_7^+}} (\mathbb{R}\Psi_0 \oplus \Psi_0^\perp \oplus W) \\ &= (\mathcal{M} \times \mathbb{R}\Psi_0) \oplus \Psi^\perp \oplus T\mathcal{M} \end{aligned}$$

and Ψ is the constant section of $\mathcal{M} \times \mathbb{R}\Psi_0$ corresponding to Ψ_0. Note that Theorem D.0.39 tells us that the torsion of the $Spin_7^+$-structure can be identified with $\nabla\Psi$. The torsion[3] $\nabla\Psi$ is a function on $\mathcal{M}$ with values in the associated bundle

$$T\mathcal{M} \otimes S\mathcal{M}^+ = B_{Spin_7^+} \times_{\widetilde{Spin_7^+}} (W \otimes \Delta^+)$$

which splits into irreducible subbundles corresponding to the irreducible summands of the $\widetilde{Spin_7^+}$-representation $W \otimes \Delta^+$. As a $Spin_8$-representation

$$W \otimes \Delta^+ = \Delta^- \oplus \mathcal{C}^+$$

[3]See Appendix D for details on torsion.

where $\mathcal{C}^+$ is the kernel of Clifford multiplication and the orthogonal complement of Δ^- in $W \otimes \Delta^+$. As with the form case we omit the details of the equivariant maps which split the exact sequences resulting from the projections onto the two summands. Returning to the bundle picture, we can apply this decomposition to $\nabla\Psi$ and this gives us

$$\nabla\Psi = -\frac{1}{8} g \cdot \mathcal{D}\Psi + \mathcal{T}\Psi$$

where $\cdot$ is Clifford multiplication (i.e. $\mathfrak{R}$ and the action of $\mathfrak{o}$), so $g \cdot \mathcal{D}\Psi = \sum_{\mathfrak{i}=\mathfrak{o}}^{7} \mathfrak{i} \otimes (\mathfrak{i} \cdot \mathcal{D}\Psi)$, $\mathcal{D}$ is the Dirac operator and $\mathcal{T}$ is the twistor operator of spinors.
Now when we restrict our representation to $\widetilde{Spin_7^+} \subset Spin_8$, $\nabla\Psi$ will split into more pieces. Rather than continue with $\Delta^- \oplus \mathcal{C}^+$, note instead that, under $\widetilde{Spin_7^+}$,

$$\begin{aligned} W \otimes \Delta^+ &= W \otimes (\mathbb{R}\Psi_0 \oplus \Psi_0^\perp) \\ &= (W \otimes \mathbb{R}\Psi_0) \oplus (W \otimes \Psi_0^\perp) \end{aligned}$$

and we need to decompose $W \otimes \Psi_0^\perp$. Clifford multiplication gives

$$W \otimes \Psi_0^\perp \to W \cdot \Psi_0^\perp = \Delta^- = W$$

and the kernel of Clifford multiplication in $W \otimes \Psi_0^\perp$ is the 48-dimensional irreducible representation F of $Spin_7^+$. So, under $Spin_7^+$,

$$W \otimes \Delta^+ = (W \otimes \mathbb{R}\Psi_0) \oplus W \cdot \Psi_0^\perp \oplus F \ .$$

where in this particular case, F should be considered as a subspace of $W \otimes \Delta^+$. The reason we didn't just write W for the first two summands is that we will now be able to see which summands correspond to Δ^- and $\mathcal{C}^+$ in the $Spin_8$ decomposition.

Lemma 8.3.1. *Using Clifford multiplication $W \otimes \Delta^+ \to \Delta^-$,*

$$\begin{aligned} W \otimes \mathbb{R}\Psi_0 &\xrightarrow{\sim} W \cdot \Psi_0 = \Delta^- \\ W \cdot \Psi_0^\perp &= \Delta^- \\ F &\to 0 \end{aligned}$$

Proof. Calculation using Proposition 8.1.10. □

This shows that

$$\Delta^- = \mathrm{Diag}\big[(W \otimes \mathbb{R}\Psi_0) \oplus (W \cdot \Psi_0^\perp)\big]$$

and

$$\mathcal{C}^+ = F \oplus \mathrm{Antidiag}\big[(W \otimes \mathbb{R}\Psi_0) \oplus (W \cdot \Psi_0^\perp)\big]$$

where

$$\mathrm{Diag}\big[(W \otimes \mathbb{R}\Psi_0) \oplus (W \cdot \Psi_0^\perp)\big] \cong \mathrm{Antidiag}\big[(W \otimes \mathbb{R}\Psi_0) \oplus (W \cdot \Psi_0^\perp)\big] \cong W$$

as representations of $\widetilde{Spin_7^+}$. In total

$$-\frac{1}{8} g \cdot \mathcal{D}\Psi = \big[-\frac{1}{8} g \cdot \mathcal{D}\Psi\big]_{\mathrm{Diag}\left[(W \otimes \mathbb{R}\Psi_0) \oplus (W \cdot \Psi_0^\perp)\right]} \ ,$$

so the Dirac operator does not decompose further, and

$$\mathcal{T}\varphi = \big[\mathcal{T}\Psi\big]_F + \big[\mathcal{T}\Psi\big]_{\mathrm{Antidiag}\left[(W\otimes\mathbb{R}\Psi_0)\oplus(W\cdot\Psi_0^{\perp})\right]} \,.$$

Lemma 8.3.2. *The W-components (the diagonal and antidiagonal pieces respectively) of $-\frac{1}{8}g\cdot\mathcal{D}\Psi$ and $\mathcal{T}\Psi$ are related by*

$$\big[-\frac{1}{8}g\cdot\mathcal{D}\Psi\big]_{W\otimes\mathbb{R}\Psi_0} = -\big[\mathcal{T}\Psi\big]_{W\otimes\mathbb{R}\Psi_0}$$

and

$$\big[-\frac{1}{8}g\cdot\mathcal{D}\Psi\big]_{W\cdot\Psi_0^{\perp}} = \big[\mathcal{T}\Psi\big]_{W\cdot\Psi_0^{\perp}} \,.$$

Proof. This is very similar to the G_2 case. □

From here on we shall use a subscript W rather than the big diagonal and antidiagonal subscripts. We can prove a spinorial analogue of Theorem 8.2.4 of David.

Theorem 8.3.3. *If the canonical spinor field of a $Spin_7^+$-structure Ψ is a conformal-Killing spinor $\mathcal{T}\Psi = 0$ then furthermore Ψ is parallel.*

Proof. This is immediate from Lemma 8.3.2. □

Just as for the G_2 case we can classify $Spin_7^+$-structures in a new way, in terms of the spinor field Ψ.

Theorem 8.3.4. *$Spin_7^+$-structures can be classified into **spin-classes** as follows.*

Spin-class	*Defining relations*
0	$\nabla\Psi = 0$
W	$\big[\mathcal{T}\Psi\big]_F = 0$
F	$\nabla\Psi = \big[\mathcal{T}\Psi\big]_F$
$W\oplus F$	*No relation*

Proof. Clear from decomposition. □

We can ask the question again: does the spin-class of a $Spin_7^+$-structure equal its form-class?

8.4 The squaring map

Recall that the real spin representation $\Delta^+\oplus\Delta^-$ is reducible.

Definition 8.4.1. *The **squaring map** is the homomorphism of representations*

$$(\Delta^+\oplus\Delta^-)\otimes(\Delta^+\oplus\Delta^-)\to\Lambda W$$

given by

$$(\chi_1 + \xi_1) \otimes (\chi_2 + \xi_2) \to g\big((\cdot) \cdot (\chi_1 + \xi_1), (\chi_2 + \xi_2)\big)$$

$$\begin{aligned} &= g\big((\cdot) \cdot \chi_1, \chi_2\big) + g\big((\cdot) \cdot \chi_1, \xi_2\big) \\ &\quad + g\big((\cdot) \cdot \xi_1, \chi_2\big) + g\big((\cdot) \cdot \xi_1, \xi_2\big) \end{aligned}$$

where $(\cdot)$ indicates the position of the argument of the form.

We can use this to 'square' Ψ_0 to a form, and the same reasoning as in the G_2 case shows that the k-form components vanish for $k = 1, 2, 5, 6$. We can also spot that $\Delta^+ \oplus \Delta^-$ is an orthogonal splitting and the action of odd elements swaps Δ^+ and Δ^-. This means that, since the χ's live in Δ^+ and the ξ's live in Δ^-, the first and last of the four terms above vanish when k is odd. Similarly the remaining pair of terms vanishes when k is even. So, since $\Psi_0 \in \Delta^+$, the k-form part of the square of Ψ_0 vanishes when k is odd. This leaves $k = 0, 4, 8$.

Proposition 8.4.2. *The square of Ψ_0 is given by*

$$\Psi_0 \otimes \Psi_0 \to 1 + 4!\Phi_0 + 8!\omega_8$$

where ω_8 is the volume form in eight dimensions.

Proof. Similar to G_2 case. □

8.5 Covariant derivative formulae and type comparisons

Consider the covariant derivative $\nabla_U \Psi$ of the canonical spinor field Ψ of $B_{Spin_7^+}$ with respect to a vector field U. Since Ψ has constant norm, $g(\nabla_U \Psi, \Psi) = 0 \ \forall U$ and so $\nabla_U \Psi \in \Gamma(\Psi^\perp)$. By Proposition 8.1.10 we can always write

$$\nabla_U \Psi = A_U \cdot \Psi$$

where A_U is a 2-form or a 4-form on $\mathcal{M}$, depending linearly on U. When A_U is a 2-form we can assume the component in $\mathfrak{spin}_7^+ \subset \Lambda^2 W$ vanishes, since this part annihilates Ψ. Similarly when A_U is a 4-form, we can assume the components in any of the subspaces

$$\mathbb{R}\Phi_0 \ , \ S_0^2 \Psi_0^\perp \ , \ \Lambda_-^4 W \subset \Lambda^4 W$$

vanish. The $\mathbb{R}\Phi_0$ part vanishes for a slightly different reason to the others; as already noted, $g(\nabla\Psi, \Psi) = 0$ and $\Phi_0 \cdot \Psi_0 \in \mathbb{R}\Psi_0$. We shall write

$$\nabla\Psi = A \cdot \Psi$$

where A is a 1-form with values in the 2-forms or the 4-forms (but we must remember that the first argument of A is *not* included in the Clifford multiplication here). Then we can regard A as being a section of $T\mathcal{M} \otimes \Psi^\perp$, which is the bundle associated to

$$W \otimes \Psi_0^\perp = W \oplus F$$

as described earlier.

Definition 8.5.1. *The **type** of A is given by the summands of $W \oplus F$ present in $A \cdot \Psi$. When we write $\nabla\Psi = A \cdot \Psi$ it is obvious that the type of A is equal to the spin-class of $B_{Spin_7^+}$.*

We can say how the parts of A relate to the parts of $\nabla\Psi$:

$$\begin{aligned} A^W \cdot \Psi &= \big[-\frac{1}{8} g \cdot \mathcal{D}\Psi\big]_W + \big[\mathcal{T}\Psi\big]_W \ , \\ A^F \cdot \Psi &= \big[\mathcal{T}\varphi\big]_F \ . \end{aligned}$$

Consider the equation $\nabla_U\Psi = A_U \cdot \Psi$ where A_U is a 2-form or a 4-form. Can we derive a formula for $\nabla_U\Phi$ in terms of A_U?

Theorem 8.5.2. *Let $\nabla_U\Psi = A_U \cdot \Psi$ be satisfied.*

1. *If A_U is a 2-form,*

$$\nabla_U\Phi(X \wedge Y \wedge Z \wedge T) = -4\Phi\big(A_{U*}(X \wedge Y \wedge Z \wedge T)\big) \ ,$$

i.e.

$$\nabla_U\Phi = 4A_{U*}\Phi$$

where A_{U*} is the Lie algebra action of A_U (which is minus the pullback action $A_U{}^*$ since A_U is skew).

2. *If A_U is a 4-form,*

$$\begin{aligned} \nabla_U\Phi(X \wedge Y \wedge Z \wedge T) &= 360\omega_8(A_U \wedge X \wedge Y \wedge Z \wedge T) \\ &\quad - 36\Phi\big(A_U \star_{1,2} (X \wedge Y \wedge Z \wedge T)\big) + A_U(X \wedge Y \wedge Z \wedge T) \ , \end{aligned}$$

i.e.

$$\nabla_U\Phi = 360\iota_{A_U}\omega_8 - 36\Phi \circ A_U \star_{1,2} + A_U$$

where ω_8 is the volume form in eight dimensions and $A_U\star_{1,2}$ is contraction over the first two indices with those of A_U.

Proof. Well,

$$\begin{aligned} \nabla_U\Phi(X \wedge Y \wedge Z \wedge T) &= \frac{1}{4!}\nabla_U g(\mathfrak{R}(\cdot)\Psi, \Psi)(X \wedge Y \wedge Z \wedge T) \\ &= \frac{1}{4!} g(\mathfrak{R}(X \wedge Y \wedge Z \wedge T)\nabla_U\Psi, \Psi) + \frac{1}{4!} g(\mathfrak{R}(X \wedge Y \wedge Z \wedge T)\Psi, \nabla_U\Psi) \\ &= \frac{1}{4!} g(\mathfrak{R}(X \wedge Y \wedge Z \wedge T)\mathfrak{R}(A_U)\Psi, \Psi) + \frac{1}{4!} g(\mathfrak{R}(X \wedge Y \wedge Z \wedge T)\Psi, \mathfrak{R}(A_U)\Psi) \ . \end{aligned}$$

Now if A_U is a 2-form, when we move the $\mathfrak{R}(A_U)$ to the other side we pick up a minus sign. If A_U is a 4-form we get a plus sign. Thus

$$\nabla_U\Phi(X \wedge Y \wedge Z \wedge T) = -\frac{1}{4!} g([\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z \wedge T)]\Psi, \Psi)$$

for A_U a 2-form and where $[\cdot, \cdot]$ is the commutator. Also

$$\nabla_U\Phi(X \wedge Y \wedge Z \wedge T) = \frac{1}{4!} g(\{\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z \wedge T)\}\Psi, \Psi)$$

when A_U is a 4-form and where $\{\cdot,\cdot\}$ is the anticommutator. Lemma B.0.22 implies

$$[\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z \wedge T)] = 4\mathfrak{R}\big((A_U)_*(X \wedge Y \wedge Z \wedge T)\big)$$

for a 2-form (where A_U acts by its Lie algebra action), and

$$\begin{aligned}\{\mathfrak{R}(A_U), \mathfrak{R}(X \wedge Y \wedge Z \wedge T)\} &= 2\mathfrak{R}\big(A_U \wedge X \wedge Y \wedge Z \wedge T\big)\\ &- 36\mathfrak{R}\big((A_U) \star_{1,2} (X \wedge Y \wedge Z \wedge T)\big) + 4!g(A_U, X \wedge Y \wedge Z \wedge T)\end{aligned}$$

for a 4-form, where $\star_{1,2}$ is contraction over the first two indices of the two 4-forms. Apply Proposition 8.4.2 to get the formulae. □

Theorem 8.5.3. *The form-class of a $Spin_7^+$-structure is equal to its spin-class.*

Proof. Recall that A is a section of the bundle associated to $W \otimes \Psi_0^\perp \subset W \otimes \Lambda^2 W$. Since $*_8 d\Phi$ is proportional to $\delta\Phi$ we need look only at $\delta\Phi$, and Theorem 8.5.2 tells us

$$\delta\Phi = \sum_{\mathfrak{i}=0}^{7} \iota_\mathfrak{i} \nabla_\mathfrak{i} \Phi = 4\sum_{\mathfrak{i}=0}^{7} (A_\mathfrak{i}\Phi)(\mathfrak{i}) \ .$$

A calculation shows that for α a 2-form of type $\Psi_0^\perp$ and $u \in W$

$$2\alpha(\Phi_0)(u) = -2\Phi_0\big(\alpha(u)\big) + u \wedge \alpha \ .$$

So

$$\delta\Phi = -4\Phi\Big(\sum_{\mathfrak{i}=0}^{7} A_\mathfrak{i}\mathfrak{i}\Big) + 2\sum_{\mathfrak{i}=0}^{7} \mathfrak{i} \wedge A_\mathfrak{i} \ .$$

The first term on the right is a 3-form of type W. Consider the map

$$W \otimes \Psi_0^\perp \to W \ : \ A \to \sum_{\mathfrak{i}=0}^{7} A_\mathfrak{i}\mathfrak{i} \ .$$

This is $Spin_7^+$-equivariant and since

$$\mathfrak{1} \otimes (\mathfrak{0} \wedge \mathfrak{1} - \mathfrak{2} \wedge \mathfrak{3} - \mathfrak{4} \wedge \mathfrak{5} + \mathfrak{6} \wedge \mathfrak{7}) \to (\mathfrak{0} \wedge \mathfrak{1} - \mathfrak{2} \wedge \mathfrak{3} - \mathfrak{4} \wedge \mathfrak{5} + \mathfrak{6} \wedge \mathfrak{7})(\mathfrak{1}) = -\frac{1}{2}\mathfrak{0} \neq 0$$

the map must be a surjective homomorphism of representations. The image of A must be proportional to the W-part of A. Antisymmetrisation

$$W \otimes \Lambda^2 W \supset W \otimes \Psi_0^\perp \to \Lambda^3 W$$

sends a basis of $W \otimes \Psi_0^\perp$ (using the basis of $\Psi_0^\perp \subset \Lambda^2 W$ we know) to a basis of $\Lambda^3 W$, so we get an isomorphism of representations $W \otimes \Psi_0^\perp \to \Lambda^3 W$ (see Appendix D for a deeper perspective on this map). The second term on the right in our expression for $\delta\Phi$ is twice the image of A under this map, so contains both the W and F components of A and nothing more. In total, we can write

$$\delta\Phi = -4\Phi(\mu A^W) + 2Anti(A)$$

for some real number μ and where A^W is the W-part of A. If A has type F then $A^W = 0$ and

$Anti(A)$ has type F, so $\delta\Phi$ does too. If A has a W-part then so does $\delta\Phi$, i.e. the W-parts of the two terms do not cancel (a calculation to determine μ shows this). This proves the theorem. $\square$

Chapter 9

Special Submanifolds

Harvey and Lawson reflect in [HL82] that there are three common ways in which a geometric structure on a space can be formulated. The first is to specify an atlas of charts whose transition functions lie in a particular pseudogroup of local diffeomorphisms of $\mathbb{R}^n$. The second is to specify a distinguished family of tensor fields, and this is the viewpoint we have so far taken on G_2 and $Spin_7^+$-structures. The third way Harvey and Lawson mention[1] is to define a geometric structure by a distinguished family of subspaces. Of course, it is possible to change between these different perspectives and it is often fruitful to do so. This chapter considers the distinguished submanifolds that arise from G_2 and $Spin_7^+$-structures, in particular their mean curvature vectors.

9.1 Associative submanifolds

A real subspace of a complex vector space is said to be complex if it inherits the complex structure J from its ambient space, that is, if it is J-invariant. The same criterion can be applied to the tangent bundle of a real submanifold of a complex manifold, and if satisfied we say the submanifold is complex. Since a complex structure is a 1-fold cross product (this is explained in Appendix C) and a G_2-structure gives us a 2-fold cross product, we can transfer this idea over to our situation.

Definition 9.1.1. *A proper subspace of $\mathbb{R}^7$ is called **associative** if the restriction of the cross product on $\mathbb{R}^7$ to it is a cross product.*

Consideration of Theorem C.0.28 shows that any associative subspace of $\mathbb{R}^7$ is necessarily three-dimensional. This can be applied to submanifolds of a manifold $\mathcal{M}$ with G_2-structure.

Definition 9.1.2. *A submanifold $\mathcal{A}$ of $\mathcal{M}$ is called **associative** if the restriction of the cross product on $\mathcal{M}$ to $\mathcal{A}$ is a cross product.*

Such submanifolds are three-dimensional. It is known from the theory of calibrations[2] that for a torsion-free[3] G_2-structure, associated submanifolds are minimal. In fact they are volume-minimising in their homology classes, as are complex submanifolds of a Kähler manifold. Since 6ϕ is a calibration whenever $d\phi = 0$ this works not only for torsion-free G_2-structures but also for those of type $\mathfrak{g}_2$.

[1] In fact they say this is "more in the spirit of classical geometries".
[2] See Harvey and Lawson [HL82] for details.
[3] See Appendix D for an explanation of torsion.

Remark 9.1.3. *The local existence of associative submanifolds when the G_2-structure has any type other than 0, $\mathbb{R}$ or $\mathfrak{g}_2$ has not been proven. In the cases 0 and $\mathfrak{g}_2$ the associative submanifolds are precisely the calibrated submanifolds of the calibration φ and their existence is shown in [HL82]. The nearly-parallel type $\mathbb{R}$ is discussed in [Lot10]. The proofs use the Cartan-Kähler Theorem and the author suspects (after correspondence with J. Lotay) this proof may be used to prove local existence of associative submanifolds for type S_0^2V (i.e. when φ is coclosed) and possibly some of the non-basic types. We do not show this here; in this chapter we suppose associative submanifolds exist and derive some properties.*

It has been shown (see [Lot10]) that associative submanifolds of G_2-structures of type $\mathbb{R}$ are also minimal. This is more interesting as in this case we do not have a calibration on $\mathcal{M}$. The proof works roughly as follows. Given a manifold $\mathcal{M}$ with G_2-structure, one can construct on the Riemannian cone over $\mathcal{M}$ a $Spin_7^+$-structure (using Proposition 8.1.6 along with Proposition B.0.25). The types of these structures may be compared, and the $Spin_7^+$-structure is parallel if and only if the G_2-structure has type $\mathbb{R}$. A manifold with parallel $Spin_7^+$-structure is calibrated, and so its Cayley submanifolds (see next section) are volume-minimising in their homology classes and therefore minimal. Given any three-dimensional submanifold $\mathcal{A}$ of $\mathcal{M}$, its Riemannian cone is Cayley if and only if $\mathcal{A}$ is associative. It is well-known that the second fundamental form of the cone of $\mathcal{A}$ inside the cone of $\mathcal{M}$ has the same trace as that of the inclusion $\mathcal{A} \subset \mathcal{M}$. Thus $\mathcal{A}$ is minimal whenever it is associative.

What about minimality of associative submanifolds of the other types of G_2-structure? The proof outlined above will not work because the $Spin_7^+$-structure constructed isn't parallel in the other cases. It may be possible to construct a different kind of structure on a bigger space (see the ideas in Chapter 11) and try to use a similar method, but there should be a simpler way.

Indeed, by direct analogy one can adapt the local proof of minimality (as can be found as a note in [KN69]) in the complex case to our needs. We adopt the cross product notation (see Appendix C) for what follows:

$$X \times Y \stackrel{\text{def}}{=} 6\phi(X \wedge Y)$$

and follow the proof as analogously as we can. The first step is to derive an equation satisfied by the second fundamental form α of our associative submanifold $\mathcal{A}$ of $\mathcal{M}$. In the complex case,

$$\alpha(JX, Y) = \alpha(X, JY)$$

holds and is needed for the minimality proof. Following the proof [KN69], we begin with

$$\begin{aligned}\nabla_Z(X \times Y) &= \nabla_Z\big[6\phi(X \wedge Y)\big] \\ &= 6(\nabla_Z\phi)(X \wedge Y) + 6\phi(\nabla_Z X \wedge Y + X \wedge \nabla_Z Y)\end{aligned}$$

where ∇ is the covariant derivative of $\mathcal{M}$ and X, Y, Z are vector fields tangent to $\mathcal{A}$. Since $\mathcal{A}$ is associative, $X \times Y$ is tangent to $\mathcal{A}$ and so we can write

$$\nabla_Z(X \times Y) = \nabla_Z^{\mathcal{A}}(X \times Y) + \alpha(X \times Y, Z)$$

where $\nabla^{\mathcal{A}}$ is the covariant derivative of the submanifold $\mathcal{A}$. If we denote the restriction of ϕ to

$T\mathcal{A}$ by $\phi|_{\mathcal{A}}$ then

$$\begin{aligned}\nabla^{\mathcal{A}}_Z(X\times Y) &= \nabla^{\mathcal{A}}_Z\big[6\phi|_{\mathcal{A}}(X\wedge Y)\big]\\ &= 6(\nabla^{\mathcal{A}}_Z\phi|_{\mathcal{A}})(X\wedge Y)+6\phi|_{\mathcal{A}}(\nabla^{\mathcal{A}}_Z X\wedge Y + X\wedge \nabla^{\mathcal{A}}_Z Y)\\ &= 6\phi|_{\mathcal{A}}(\nabla^{\mathcal{A}}_Z X\wedge Y + X\wedge \nabla^{\mathcal{A}}_Z Y)\end{aligned}$$

because, by associativity, $\phi|_{\mathcal{A}}$ is the volume form on $\mathcal{A}$ and is therefore parallel along $\mathcal{A}$. Upon restriction to $\mathcal{A}$,

$$\begin{aligned}\phi(\nabla_Z X\wedge Y + X\wedge\nabla_Z Y) = \phi|_{\mathcal{A}}(\nabla^{\mathcal{A}}_Z X\wedge Y + X\wedge \nabla^{\mathcal{A}}_Z Y)&\\ + \phi(\alpha(Z,X)\wedge Y + X\wedge\alpha(Z,Y))&\end{aligned}$$

and by equating the two expressions for $\nabla_Z(X\times Y)$ we get

Lemma 9.1.4. *For α the second fundamental form of an associative submanifold $\mathcal{A}$ of a manifold $\mathcal{M}$ with G_2-structure,*

$$\alpha(X\times Y, Z) = 6(\nabla_Z\phi)(X\wedge Y) + \alpha(X,Z)\times Y + X\times\alpha(Y,Z)$$

where $\times$ is the cross product given by the G_2-structure.

This is our analogue of the formula $\alpha(JX,Y)=\alpha(X,JY)$ for a 1-fold cross product J, except with an extra term due to the torsion of the G_2-structure. The next thing we need to do is use this formula to get a formula satisfied by the Weingarten map A_η of the submanifold $\mathcal{A}$ given with respect to a unit normal vector field η. Recall that the Weingarten map, or shape operator, is just the second fundamental form with an index raised:

$$g(A_\eta X, Y) \stackrel{\text{def}}{=} g(\alpha(X,Y),\eta)\ .$$

Lemma 9.1.5. *For X,Y,Z vector fields tangent to $\mathcal{A}$ and A_η the Weingarten map of $\mathcal{A}$ where η is a locally-defined unit normal,*

$$\begin{aligned}&g(A_\eta X\times Y, Z) + g(X\times A_\eta Y, Z)\\ &\qquad = -g(A_\eta(X\times Y),Z) + 24d\phi(X\wedge Y\wedge Z\wedge\eta)\\ &\qquad\qquad + 6(\nabla_\eta\phi)(X\wedge Y\wedge Z)\ .\end{aligned}$$

Proof. By rearranging each of the terms $g(A_\eta X\times Y, Z)$, $g(X\times A_\eta Y, Z)$ and $g(A_\eta(X\times Y),Z)$ and applying Lemma 9.1.4 one finds three terms which nearly give the antisymmetrisation of $\nabla\phi$. The final term is the required correction. $\square$

Although it may not look so, Lemma 9.1.5 is the direct analogue of the condition

$$A_\eta J = -JA_\eta$$

where J is the complex structure on a complex submanifold of a Kähler manifold, except with extra terms due to the torsion of the G_2-structure.

Lemma 9.1.6. *Let $\mathcal{A}$ be an associative submanifold of the manifold $\mathcal{M}$, where $\mathcal{M}$ possesses a*

G_2-structure. The mean curvature vector of $\mathcal{A}$ is

$$H = 8d\phi\big(\phi|_{\mathcal{A}}\big) \ .$$

Proof. Since $\mathcal{A}$ is associative, we can choose a local Cayley frame $1, \ldots, 7$ which is adapted to $\mathcal{A}$, so that $1, 2, 3$ span the tangent spaces of $\mathcal{A}$ and the orientation is given by $1 \wedge 2 \wedge 3$. We begin by calculating the trace of the Weingarten map A_η:

$$\mathrm{tr} A_\eta = g(A_\eta 1, 1) + g(A_\eta 2, 2) + g(A_\eta 3, 3) \ .$$

Now $2 \times 3 = 1$ so by Lemma 9.1.5

$$\begin{aligned} g(A_\eta 1, 1) &= g(A_\eta (2 \times 3), 1) \\ &= -g(A_\eta 2 \times 3, 1) - g(2 \times A_\eta 3, 1) + 24 d\phi(1 \wedge 2 \wedge 3 \wedge \eta) \\ &\qquad + 6(\nabla_\eta \phi)(1 \wedge 2 \wedge 3) \ . \end{aligned}$$

But

$$\begin{aligned} \nabla\phi = \ & \nabla 1 \wedge 2 \wedge 3 + 1 \wedge \nabla 2 \wedge 3 + 1 \wedge 2 \wedge \nabla 3 \\ & + \cdots \\ & - \nabla 3 \wedge 5 \wedge 6 - 3 \wedge \nabla 5 \wedge 6 - 3 \wedge 5 \wedge \nabla 6 \end{aligned}$$

and the contraction of the last three indices of this with $1 \wedge 2 \wedge 3$ is zero because ∇i is orthogonal to i, so

$$(\nabla_\eta \phi)(1 \wedge 2 \wedge 3) = 0 \ .$$

As a side note, this implies that

$$d\phi\big(\phi|_{\mathcal{A}}\big) = \mathcal{T}\phi\big(\phi|_{\mathcal{A}}\big)$$

where $\mathcal{T}$ is the twistor operator and we can write the mean curvature like this if we wish. Continuing the calculation,

$$\begin{aligned} \mathrm{tr} A_\eta = & - g(A_\eta 2 \times 3, 1) - g(2 \times A_\eta 3, 1) \\ & - g(A_\eta 3 \times 1, 2) - g(3 \times A_\eta 1, 2) \\ & - g(A_\eta 1 \times 2, 3) - g(1 \times A_\eta 2, 3) \\ & + 72 d\phi(1 \wedge 2 \wedge 3 \wedge \eta) \ . \end{aligned}$$

But ϕ, and hence the cross product, is skew, which means

$$g(2 \times A_\eta 3, 1) = g(A_\eta 3 \times 1, 2) \ .$$

Also

$$\begin{aligned} g(A_\eta 2 \times 3, 1) &= g(A_\eta 2 \times 3, 2 \times 3) \\ &= g(A_\eta 2, 2) \end{aligned}$$

which follows from B.0.20. Adding all the terms gives

$$\mathrm{tr}A_\eta = -2\mathrm{tr}A_\eta + 72 d\phi(1 \wedge 2 \wedge 3 \wedge \eta) \ .$$

Note that $1 \wedge 2 \wedge 3$ is the volume form of $\mathcal{A}$, so is equal to $\phi|_{\mathcal{A}}$. We can rewrite the above equation as

$$H = 8 d\phi\big(\phi|_{\mathcal{A}}\big) \ .$$

□

Recall Definition 7.2.8, that a manifold with G_2-structure has a naturally defined vector field θ, defined to be the V-part of $d\phi$:

$$\theta \stackrel{\text{def}}{=} -24 d\phi(\phi) \ .$$

Also recall Definition 7.2.10, that a manifold with G_2-structure has a naturally defined symmetric-traceless tensor field of rank two, denoted T. Now we can calculate the mean curvature vector of any associative submanifold.

Theorem 9.1.7. *Let $\mathcal{A}$ be an associative submanifold of the manifold $\mathcal{M}$, where $\mathcal{M}$ possesses a G_2-structure. The mean curvature vector of $\mathcal{A}$ is*

$$H = -\frac{1}{12}\theta^{\perp\mathcal{A}} + 2\phi(T_{T\mathcal{A}\to N\mathcal{A}})$$

where $\theta^{\perp\mathcal{A}}$ is the component of the vector field θ orthogonal to $\mathcal{A}$ and $T_{T\mathcal{A}\to N\mathcal{A}}$ is the component of the symmetric-traceless rank two tensor field T sending $T\mathcal{A}$ to the normal bundle $N\mathcal{A}$.

Proof. Using Lemma 9.1.6, we can calculate the contribution to H from each of the basic parts of $d\phi$. The full result then follows by linearity. We use a Cayley frame $1, \dots, 7$ as in the previous lemma. The $\mathfrak{g}_2$-part is not present in $d\phi$ and so makes no contribution. For the $\mathbb{R}$-part, suppose $d\phi$ is proportional to $*\phi$:

$$*\phi(1 \wedge 2 \wedge 3) = 0 \ .$$

For the V-part, suppose $d\phi = \frac{1}{4}\theta \wedge \phi$.

$$\begin{aligned}
(\theta \wedge \phi)(1 \wedge 2 \wedge 3) &= \iota_3\iota_2\iota_1 \theta \wedge \phi \\
&= \iota_3\iota_2\Big[\frac{1}{4}\theta(1)\phi - \frac{3}{4}\theta \wedge \phi(1)\Big] \\
&= \frac{1}{4}\theta(1)\phi(2 \wedge 3) - \frac{3}{4}\iota_3\iota_2\theta \wedge \phi(1) \\
&= \frac{1}{4}\theta(1)\phi(2 \wedge 3) - \frac{3}{4}\iota_3\Big[\frac{1}{3}\theta(2)\phi(1) - \frac{2}{3}\theta \wedge \phi(1 \wedge 2)\Big] \\
&= \frac{1}{4}\theta(1)\phi(2 \wedge 3) + \frac{1}{4}\theta(2)\phi(3 \wedge 1) \\
&+ \frac{1}{4}\theta(3)\phi(1 \wedge 2) - \frac{1}{4}\theta \wedge \phi(1 \wedge 2 \wedge 3) \\
&= \frac{1}{4!}\big[\theta(1)1 + \theta(2)2 + \theta(3)3 - \theta\big] \\
&= -\frac{1}{4!}\theta^{\perp\mathcal{A}} \ .
\end{aligned}$$

So the contribution to H from the V-part is

$$8d\phi(1 \wedge 2 \wedge 3) = -\frac{1}{12}\theta^{\perp\mathcal{A}} .$$

Finally, for the contribution from the S_0^2V-part, assume $d\phi = -T(*\phi)$ for T the canonical section of $S_0^2T\mathcal{M}$. With respect to the splitting $T\mathcal{M} = T\mathcal{A} \oplus N\mathcal{A}$ we can write

$$T = \begin{pmatrix} T|_{\mathcal{A}} & T_{N\mathcal{A}\to T\mathcal{A}} \\ T_{T\mathcal{A}\to N\mathcal{A}} & T|_{N\mathcal{A}} \end{pmatrix} .$$

With respect to the frame $1, \ldots, 7$ we can write

$$T = \sum_{i=1}^{7}\sum_{j=1}^{7} T^{ij} i \otimes j$$

where $T^{ji} = T^{ij}$ and

$$T_{T\mathcal{A}\to N\mathcal{A}} = \sum_{i=1}^{3}\sum_{j=4}^{7} T^{ij} i \otimes j .$$

We have

$$\begin{aligned}
T(*\phi)(1 \wedge 2 \wedge 3) = \sum_{i=1}^{7}\sum_{j=1}^{7} T^{ij}\big[&g(i,4) j \wedge 5 \wedge 6 \wedge 7 + g(i,5) 4 \wedge j \wedge 6 \wedge 7 \\
&+ \cdots \\
&- g(i,1) j \wedge 2 \wedge 4 \wedge 7 - g(i,2) 1 \wedge j \wedge 4 \wedge 7 \\
&- g(i,4) 1 \wedge 2 \wedge j \wedge 7 - g(i,7) 1 \wedge 2 \wedge 4 \wedge j\big](1 \wedge 2 \wedge 3) \\
= \sum_{i=1}^{7}\sum_{j=1}^{7} T^{ij}\big[&g(i,6) 2 \wedge 3 \wedge j \wedge 7 + g(i,7) 2 \wedge 3 \wedge 6 \wedge j \\
&- g(i,4) 2 \wedge 3 \wedge j \wedge 5 - g(i,5) 2 \wedge 3 \wedge 4 \wedge j \\
&+ g(i,5) 1 \wedge 3 \wedge j \wedge 7 + g(i,7) 1 \wedge 3 \wedge 5 \wedge j \\
&+ g(i,4) 1 \wedge 3 \wedge j \wedge 6 + g(i,6) 1 \wedge 3 \wedge 4 \wedge j \\
&+ g(i,5) 1 \wedge 2 \wedge j \wedge 6 + g(i,6) 1 \wedge 2 \wedge 5 \wedge j \\
&- g(i,4) 1 \wedge 2 \wedge j \wedge 6 - g(i,7) 1 \wedge 2 \wedge 4 \wedge j\big](1 \wedge 2 \wedge 3) \\
= \frac{1}{4!}\big[&T^{61} 7 - T^{71} 6 - T^{41} 5 + T^{51} 4 \\
&- T^{52} 7 + T^{72} 5 - T^{42} 6 + T^{62} 4 \\
&+ T^{53} 6 - T^{63} 5 - T^{43} 7 + T^{73} 4\big] \\
= -\frac{1}{4}\big[&T^{16}\phi(1 \wedge 6) + T^{17}\phi(1 \wedge 7) + T^{14}\phi(1 \wedge 4) + T^{15}\phi(1 \wedge 5) \\
&+ T^{25}\phi(2 \wedge 5) + T^{27}\phi(2 \wedge 7) + T^{24}\phi(2 \wedge 4) + T^{26}\phi(2 \wedge 6) \\
&+ T^{35}\phi(3 \wedge 5) + T^{36}\phi(3 \wedge 6) + T^{34}\phi(3 \wedge 4) + T^{37}\phi(3 \wedge 7)\big] \\
= -\frac{1}{4}&\sum_{i=1}^{3}\sum_{j=4}^{7} T^{ij}\phi(i \otimes j) \\
= -\frac{1}{4}&\phi(T_{T\mathcal{A}\to N\mathcal{A}}) .
\end{aligned}$$

The contribution to H from the S_0^2V-part is therefore

$$8d\phi(1 \wedge 2 \wedge 3) = 2\phi(T_{T\mathcal{A}\to N\mathcal{A}}) \ .$$

□

The above theorem immediately reproduces

Corollary 9.1.8. *Let $\mathcal{M}$ be a manifold with G_2-structure. If the type is 0, $\mathbb{R}$ or $\mathfrak{g}_2$ then any and all associative submanifolds of $\mathcal{M}$ are minimal.*

In general we have an equation which characterises minimality of associative submanifolds.

Corollary 9.1.9. *Let $\mathcal{M}$ be a manifold with G_2-structure. Then, an associative submanifold $\mathcal{A}$ of $\mathcal{M}$ is minimal if and only if at all points of $\mathcal{A}$, the equation*

$$\theta^{\perp\mathcal{A}} = 24\phi(T_{T\mathcal{A}\to N\mathcal{A}}) \ .$$

holds.

Proof. Follows from Theorem 9.1.7. □

9.2 Cayley submanifolds

Many of the statements about associative submanifolds have direct analogues for manifolds with $Spin_7^+$-structures. Recall that a $Spin_7^+$-structure on an 8-dimensional manifold $\mathcal{M}$ is equivalent to specifying a 3-fold cross product on $\mathcal{M}$.

Definition 9.2.1. *A proper subspace of $\mathbb{R}^8$ is called **Cayley** if the restriction of the cross product on $\mathbb{R}^8$ to it is a cross product.*

Consideration of Theorem C.0.28 shows that any Cayley subspace of $\mathbb{R}^8$ is necessarily four-dimensional. This can be applied to submanifolds of a manifold $\mathcal{M}$ with $Spin_7^+$-structure.

Definition 9.2.2. *A submanifold $\mathcal{C}$ of $\mathcal{M}$ is called **Cayley** if the restriction of the cross product on $\mathcal{M}$ to $\mathcal{C}$ is a cross product.*

Such submanifolds are four-dimensional. It is known from the theory of calibrations[4] that for a torsion-free[5] $Spin_7^+$-structure, Cayley submanifolds are minimal; moreover, they are volume-minimising in their homology classes. Since 24Φ is a calibration whenever $d\Phi = 0$ this works only for torsion-free $Spin_7^+$-structures.

Remark 9.2.3. *The local existence of Cayley submanifolds when the $Spin_7^+$-structure has any type other than 0 has not been proven. In the case 0 the Cayley submanifolds are precisely the calibrated submanifolds of the calibration Φ and their existence is shown in [HL82]. Unlike the G_2 case, as mentioned in Remark 9.1.3, we do not know about the other cases at all because Φ is coclosed if and only if it is parallel. In this chapter we suppose Cayley submanifolds exist and derive some properties.*

[4]See Harvey and Lawson [HL82] for details.
[5]See Appendix D for an explanation of torsion.

We adopt the cross product notation (see Appendix C) for what follows:

$$W \times X \times Y \stackrel{\text{def}}{=} 24\Phi(W \wedge X \wedge Y)$$

and follow the proof for associative submanifolds closely.

Lemma 9.2.4. *For α the second fundamental form of a Cayley submanifold $\mathcal{C}$ of a manifold $\mathcal{M}$ with $Spin_7^+$-structure,*

$$\begin{aligned}\alpha(W \times X \times Y, Z) &= 24(\nabla_Z \Phi)(W \wedge X \wedge Y) \\ &\quad + \alpha(W, Z) \times X \times Y + W \times \alpha(X, Z) \times Y + W \times X \times \alpha(Y, Z)\end{aligned}$$

where $\times$ is the cross product given by the $Spin_7^+$-structure.

Proof. On the one hand

$$\begin{aligned}\nabla_Z(W \times X \times Y) &= \nabla_Z\big[24\Phi(W \wedge X \wedge Y)\big] \\ &= 24(\nabla_Z \Phi)(W \wedge X \wedge Y) \\ &\quad + 24\Phi(\nabla_Z W \wedge X \wedge Y + W \wedge \nabla_Z X \wedge Y + W \wedge X \wedge \nabla_Z Y)\end{aligned}$$

where ∇ is the covariant derivative of $\mathcal{M}$ and W, X, Y, Z are vector fields tangent to $\mathcal{C}$. Upon restriction to $\mathcal{C}$,

$$\begin{aligned}&\Phi(\nabla_Z W \wedge X \wedge Y + W \wedge \nabla_Z X \wedge Y + W \wedge X \wedge \nabla_Z Y) \\ &\quad = \Phi|_{\mathcal{C}}(\nabla^{\mathcal{C}}_Z W \wedge X \wedge Y + W \wedge \nabla^{\mathcal{C}}_Z X \wedge Y + W \wedge X \wedge \nabla^{\mathcal{C}}_Z Y) \\ &\quad + \Phi(\alpha(W, Z) \wedge X \wedge Y + W \wedge \alpha(X, Z) \wedge Y + W \wedge X \wedge \alpha(Y, Z))\end{aligned}$$

where $\nabla^{\mathcal{C}}$ is the covariant derivative of $\mathcal{C}$ and $\Phi|_{\mathcal{C}}$ is the restriction of Φ to $T\mathcal{C}$. Since $\mathcal{C}$ is Cayley, $W \times X \times Y$ is tangent to $\mathcal{C}$ and we can write

$$\nabla_Z(W \times X \times Y) = \nabla^{\mathcal{C}}_Z(W \times X \times Y) + \alpha(W \times X \times Y, Z)$$

where

$$\begin{aligned}\nabla^{\mathcal{C}}_Z(W \times X \times Y) &= \nabla^{\mathcal{C}}_Z\big[24\Phi|_{\mathcal{C}}(W \wedge X \wedge Y)\big] \\ &= 24(\nabla^{\mathcal{C}}_Z \Phi|_{\mathcal{C}})(W \wedge X \wedge Y) \\ &\quad + 24\Phi|_{\mathcal{C}}(\nabla^{\mathcal{C}}_Z W \wedge X \wedge Y + W \wedge \nabla^{\mathcal{C}}_Z X \wedge Y + W \wedge X \wedge \nabla^{\mathcal{C}}_Z Y) \\ &= 24\Phi|_{\mathcal{C}}(\nabla^{\mathcal{C}}_Z W \wedge X \wedge Y + W \wedge \nabla^{\mathcal{C}}_Z X \wedge Y + W \wedge X \wedge \nabla^{\mathcal{C}}_Z Y)\end{aligned}$$

because, since $\mathcal{C}$ is Cayley, $\Phi|_{\mathcal{C}}$ is the volume form on $\mathcal{C}$ and is therefore parallel along $\mathcal{C}$. Equating the two expressions for $\nabla_Z(W \times X \times Y)$ gives the formula. □

Lemma 9.2.5. *For W, X, Y, Z vector fields tangent to $\mathcal{C}$ and A_η the Weingarten map of $\mathcal{C}$*

where η is a locally-defined unit normal,

$$\begin{aligned} g(A_\eta W\times X\times Y,Z)&+g(W\times A_\eta X\times Y,Z)+g(W\times X\times A_\eta Y,Z)\\ &=-g(A_\eta(W\times X\times Y),Z)-120d\Phi(W\wedge X\wedge Y\wedge Z\wedge\eta)\\ &\quad+24(\nabla_\eta\Phi)(W\wedge X\wedge Y\wedge Z)\ . \end{aligned}$$

Proof. By rearranging each of the terms $g(A_\eta W\times X\times Y,Z)$, $g(W\times A_\eta X\times Y,Z)$, $g(W\times X\times A_\eta Y,Z)$ and $g(A_\eta(W\times X\times Y),Z)$ and applying Lemma 9.2.4 one can easily find four terms which nearly give the antisymmetrisation of $\nabla\Phi$. The final term is the required correction. □

Lemma 9.2.6. *Let $\mathcal{C}$ be a Cayley submanifold of the manifold $\mathcal{M}$, where $\mathcal{M}$ possesses a $Spin_7^+$-structure. The mean curvature vector of $\mathcal{C}$ is*

$$H=-24d\Phi\big(\Phi|_{\mathcal{C}}\big)\ .$$

Proof. Since $\mathcal{C}$ is Cayley, we can choose a local Cayley frame $0,1,\dots,7$ which is adapted to $\mathcal{C}$, so that $0,1,2,3$ span the tangent spaces of $\mathcal{C}$ and the orientation is given by $0\wedge1\wedge2\wedge3$. We begin by calculating the trace of the Weingarten map A_η:

$$\mathrm{tr}A_\eta=g(A_\eta 0,0)+g(A_\eta 1,1)+g(A_\eta 2,2)+g(A_\eta 3,3)\ .$$

Now $1\times2\times3=-0$ so by Lemma 9.2.5

$$\begin{aligned} g(A_\eta 0,0)&=-g(A_\eta(1\times2\times3),0)\\ &=g(A_\eta 1\times2\times3,0)+g(1\times A_\eta 2\times3,0)\\ &\quad+g(1\times2\times A_\eta 3,0)-120d\Phi(0\wedge1\wedge2\wedge3\wedge\eta)\\ &\quad\quad+24(\nabla_\eta\Phi)(0\wedge1\wedge2\wedge3)\ . \end{aligned}$$

Just as for the G_2 case

$$(\nabla_\eta\Phi)(0\wedge1\wedge2\wedge3)=0$$

and as a side note this implies that

$$d\Phi\big(\Phi|_{\mathcal{C}}\big)=\mathcal{T}\Phi\big(\Phi|_{\mathcal{C}}\big)$$

where $\mathcal{T}$ is the twistor operator and we can write the mean curvature like this if we wish. Continuing the calculation,

$$\begin{aligned} \mathrm{tr}A_\eta=\ &g(A_\eta 1\times2\times3,0)+g(1\times A_\eta 2\times3,0)+g(1\times2\times A_\eta 3,0)\\ &-g(A_\eta 2\times3\times0,1)-g(2\times A_\eta 3\times0,1)-g(2\times3\times A_\eta 0,1)\\ &+g(A_\eta 3\times0\times1,2)+g(3\times A_\eta 0\times1,2)+g(3\times0\times A_\eta 1,2)\\ &-g(A_\eta 0\times1\times2,3)-g(0\times A_\eta 1\times2,3)-g(0\times1\times A_\eta 2,3)\\ &\quad-480d\Phi(0\wedge1\wedge2\wedge3\wedge\eta)\ . \end{aligned}$$

But Φ, and hence the cross product, is skew, which means

$$g(1\times A_\eta 2\times3,0)=-g(A_\eta 2\times3\times0,1)\ .$$

Also

$$\begin{aligned} g(A_\eta 1 \times 2 \times 3, 0) &= -g(A_\eta 1 \times 2 \times 3, 1 \times 2 \times 3) \\ &= -g(A_\eta 1, 1) \end{aligned}$$

which follows from Lemma B.0.21. Adding all the terms gives

$$\mathrm{tr}A_\eta = -4\mathrm{tr}A_\eta - 480 d\Phi(0 \wedge 1 \wedge 2 \wedge 3 \wedge \eta) \ .$$

Note that $0 \wedge 1 \wedge 2 \wedge 3$ is the volume form of $\mathcal{C}$, so is equal to $\Phi|_{\mathcal{C}}$. We can rewrite the above equation as

$$H = -24 d\Phi\big(\Phi|_{\mathcal{C}}\big) \ .$$

□

Recall that any $Spin_7^+$-structure has a naturally defined vector field θ, defined to be the W-part of $*d\Phi$:

$$\theta \stackrel{\text{def}}{=} -210 d\Phi(\Phi) \ .$$

Theorem 9.2.7. *Let $\mathcal{C}$ be a Cayley submanifold of the manifold $\mathcal{M}$, where $\mathcal{M}$ possesses a $Spin_7^+$-structure. The mean curvature vector of $\mathcal{C}$ is*

$$H = -\frac{1}{35}\theta^{\perp\mathcal{C}} - 24\big[d\Phi\big]_F\big(\Phi|_{\mathcal{C}}\big) \ .$$

Proof. We use the same method as in the G_2 case. For the contribution from the W-part, suppose $d\Phi = \frac{1}{7}\theta \wedge \Phi$. Then

$$\begin{aligned} (\theta \wedge \Phi)(0 \wedge 1 \wedge 2 \wedge 3) &= \iota_3\iota_2\iota_1\iota_0 \theta \wedge \Phi \\ &= \iota_3\iota_2\iota_1\big[\frac{1}{5}\theta(0)\Phi - \frac{4}{5}\theta \wedge \Phi(0)\big] \\ &= \frac{1}{5}\theta(0)\Phi(1 \wedge 2 \wedge 3) - \frac{4}{5}\iota_3\iota_2\iota_1\theta \wedge \Phi(0) \\ &= \frac{1}{5}\theta(0)\Phi(1 \wedge 2 \wedge 3) - \frac{4}{5}\iota_3\iota_2\big[\frac{1}{4}\theta(1)\Phi(0) - \frac{3}{4}\theta \wedge \Phi(0 \wedge 1)\big] \\ &= \frac{1}{5}\theta(0)\Phi(1 \wedge 2 \wedge 3) - \frac{1}{5}\theta(1)\Phi(0 \wedge 2 \wedge 3) \\ &+ \frac{3}{5}\iota_3\big[\frac{1}{3}\theta(2)\Phi(0 \wedge 1) - \frac{2}{3}\theta \wedge \Phi(0 \wedge 1 \wedge 2)\big] \\ &= \frac{1}{5}\theta(0)\Phi(1 \wedge 2 \wedge 3) - \frac{1}{5}\theta(1)\Phi(0 \wedge 2 \wedge 3) \\ &+ \frac{1}{5}\theta(2)\Phi(0 \wedge 1 \wedge 3) - \frac{1}{5}\theta(3)\Phi(0 \wedge 1 \wedge 2) \\ &+ \frac{1}{5}\theta \wedge \Phi(0 \wedge 1 \wedge 2 \wedge 3) \\ &= \frac{1}{5!}\big[-\theta(0)0 - \theta(1)1 - \theta(2)2 - \theta(3)3 + \theta\big] \\ &= \frac{1}{5!}\theta^{\perp\mathcal{C}} \ . \end{aligned}$$

So the contribution to H from the W-part is

$$-24 d\Phi(0 \wedge 1 \wedge 2 \wedge 3) = -\frac{1}{35}\theta^{\perp\mathcal{C}} \ .$$

We have not been able to find a useful expression for the contribution made by the F-part of $d\Phi$. $\square$

Chapter 10

Quotient Constructions

This chapter concerns a type of quotient of a Riemannian manifold with seven anticommuting metric-compatible complex structures. The procedure is analogous to the various quotient constructions already known: the hyper-Kähler quotient of Hitchin, Karlhede, Lindström and Roček [HKLR87]—itself a modification of the Marsden-Weinstein symplectic quotient [MW74]—and the quaternionic-Kähler quotient of Galicki and Lawson [GL88]. We begin by reviewing the hyper-Kähler case before considering our own version.

10.1 Hyper-Kähler quotients

Let (Σ^{4n}, g) be locally hyper-Kähler, i.e. a Riemannian manifold with restricted holonomy contained in Sp_n. We can choose three local parallel anticommuting complex structures J_1, J_2, J_3 on Σ. The group of isometries of (Σ, g) that also preserve each of J_1, J_2, J_3 will be denoted $\mathrm{Aut}(\Sigma)$ (it is automatic that isometries preserve the bundle spanned by J_1, J_2, J_3, but they need not preserve this trivialisation). Consider the action of a group $G < \mathrm{Aut}(\Sigma)$ on Σ, and denote by μ its moment map. That is, μ is a map $\Sigma \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ given by

$$d\big[\mu_i(\gamma)\big] = J_i(\gamma)$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ and $\gamma \in \mathfrak{g}$. We use the same symbol γ for the Lie algebra element and the vector field on Σ that it generates, which is the meaning of γ the right-hand side of the equation. A moment map μ is determined only up to constant functions $\Sigma \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$, and may not exist because $J_i(\gamma)$ may not be exact. A sufficient condition for existence is $H^1(\Sigma, \mathbb{R}) = 0$ and if this isn't the case we can work locally. According to [MW74], the moment map is always equivariant with respect to a modified action of G on $\mathfrak{g}^*$ but in many cases such as when G is compact or semisimple (or a torus [HKLR87]) we can choose μ to be equivariant with respect to the standard coadjoint action. Note that we do not distinguish between the complex structures J_1, J_2, J_3 and the corresponding symplectic forms $\omega_1, \omega_2, \omega_3$ on Σ. The map μ is smooth, so let ν be a regular value (or more generally, a *weakly regular value*—see [MW74]) in $\mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$. Define

$$\mathcal{M}_\nu \stackrel{\text{def}}{=} \mu^{-1}(\nu)$$

so that $\mathcal{M}_\nu \subset \Sigma$ is a submanifold. It is easy to show that $\mathcal{M}_\nu$ is preserved by the action of G, so we can look at the space of orbits

$$\mathcal{B}_\nu \stackrel{\text{def}}{=} \mathcal{M}_\nu / G \ .$$

Of course, $\mathcal{B}_\nu$ need not be a manifold. Sufficient conditions for $\mathcal{B}_\nu$ to be a manifold are for example that G act freely and properly on $\mathcal{M}_\nu$, which are precisely the conditions under which the projection $\mathcal{M}_\nu \to \mathcal{B}_\nu$ is a principal G-bundle. Otherwise we can restrict our attention to parts of $\mathcal{B}_\nu$ that do form a manifold (if any), or make statements in terms of orbifolds instead as is done in [GL88].

Theorem 10.1.1. *([HKLR87]) The unique Riemannian metric on $\mathcal{B}_\nu$ which makes $\mathcal{M}_\nu \to \mathcal{B}_\nu$ a Riemannian submersion is locally hyper-Kähler.*

It is true that a hyper-Kähler manifold is complex symplectic (see [Bes87]), and the proof of the above theorem makes use of the complex symplectic form with respect to one of the complex structures to show that the induced metric is Kähler with respect to that complex structure. This is not the viewpoint we shall take in our construction.

We should remark at this point that the same procedure will work to define an ordinary Kähler quotient, and this was first explained in [HKLR87]. That is, if Σ is Kähler but not hyper-Kähler then $\mathcal{B}_\nu$ naturally inherits a Kähler metric. This fact will later allow us to confirm one of our own results.

10.2 A 7-complex quotient

Our original motivation for considering a type of quotient of a Riemannian manifold with seven complex structures was to try to construct new examples of manifolds with $Spin_7$ holonomy. Whilst this has not been achieved, the situation is still interesting and may indicate ways in which that goal may be accomplished. We discuss this speculation in the next section in detail.

Let us now set the scene for our construction. We will begin with a very simple case, where Σ is $\mathbb{R}^{16}$ and the acting group G is a circle.

The real Clifford algebra $C\ell_7$ has two inequivalent irreducible real representations Δ^+ and Δ^-. We will use the space

$$\Sigma \stackrel{\text{def}}{=} \Delta^+ \oplus \Delta^+ \ .$$

The representation Δ^+ (and Δ^-) can be considered as a copy of the octonions $\mathbb{O}$ and using this, Σ inherits an inner product. A Cayley frame $\mathfrak{1}, \ldots, \mathfrak{7}$ of $V = \mathrm{Im}\mathbb{O} \subset C\ell_7$ defines on Σ seven complex structures $J_{\mathfrak{1}}, \ldots, J_{\mathfrak{7}}$ given by

$$J_{\mathfrak{i}}(x) \stackrel{\text{def}}{=} \mathfrak{i} \cdot x$$

where $\cdot$ is Clifford multiplication. These complex structures anticommute and are all compatible with the inner product, and again we make no distinction between $J_{\mathfrak{i}}$ and the corresponding element of $\Lambda^2\Sigma$. Analogously to the case of a hypercomplex structure (see [Bes87] or [Joy95]), the $J_{\mathfrak{i}}$'s are merely seven of a whole S^6's worth of complex structures on Σ.

The Schur algebra of the representation Σ is easily seen to be the real algebra of 2×2 real matrices $\mathbb{R}(2)$, because Δ^+ is irreducible. Conveniently, $\mathbb{R}(2)$ is the algebra of paraquaternions

$$\tilde{\mathbb{H}} \cong \mathbb{R}(2) \ .$$

The paraquaternions $\tilde{\mathbb{H}}$ may be described in a very similar way to the paracomplex numbers $\tilde{\mathbb{C}}$. They are generated by a basis $1, \mathscr{I}, \mathscr{J}, \mathscr{K}$ (as for the quaternions, whether or not one considers such a basis to be part of the definition is debatable—we do not) which anticommute and satisfy

$$\mathscr{I}^2 = -1 \ , \ \ \mathscr{J}^2 = \mathscr{K}^2 = 1 \ .$$

The complex structure $\mathscr{I}$ will be the generator of a U_1-action on Σ (recall the Lie algebra of U_1 is $\mathrm{Im}\mathbb{C}$).

Proposition 10.2.1. *The representation $\Sigma = \Delta^+ \oplus \Delta^+$ admits a U_1-action, defined by*

$$e^{i\theta}\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which preserves $J_1, \ldots, J_7$ (and hence all S^6 complex structures) and acts isometrically.

Proof. Notice that we can now think of Σ as a real manifold of dimension 16, with Riemannian metric and seven complex structures $J_1, \ldots, J_7$ given by their parallel extensions. Of course Σ is just Euclidean space $\mathbb{R}^{16}$. It is clear the definition is a U_1-action. Its derivative at $\theta = 0$ is the Lie algebra generator $\mathscr{I}$, which lies in the Schur algebra and hence preserves the complex structures. It is easy to check the action is isometric. □

Remark 10.2.2. *One can see that the elements $\mathscr{J}$ and $\mathscr{K}$ also generate actions which preserve $J_1, \ldots, J_7$ and these actions are not by isometries. Note that the real algebra of paracomplex numbers $\tilde{\mathbb{C}}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.*

Definition 10.2.3. *The vector field γ on Σ is that which is* ***tangent to the flow*** *of the U_1-action.*

Two things are immediate: γ has precisely one zero, at $0 \in \Sigma$, and that γ is Killing on Σ. It is not difficult to show

$$||\gamma_x|| = ||x|| \ .$$

Observe that

$$0 = \mathscr{L}_\gamma J_{\mathrm{i}} = 3\iota_\gamma dJ_{\mathrm{i}} + 2d(\iota\gamma J_{\mathrm{i}}) = 2d(J_{\mathrm{i}}\gamma)$$

so the 1-form $J_{\mathrm{i}}\gamma$ is closed and therefore exact, because $H^1(\Sigma, \mathbb{R}) = 0$. There exist functions $\mu_{\mathrm{i}} : \Sigma \to \mathbb{R}$ satisfying

$$d\mu_{\mathrm{i}} = J_{\mathrm{i}}\gamma \ , \quad \mu_{\mathrm{i}}(0) = 0 \ .$$

Definition 10.2.4. *The* ***moment map*** *$\mu : \Sigma \to V = \mathrm{Im}\mathbb{O}$ is defined as*

$$\mu \stackrel{\text{def}}{=} (\mu_1, \ldots, \mu_7)$$

with respect to the Cayley frame $1, \ldots, 7$ of V.

It immediately follows that μ is smooth and $\mu(0) = 0$.

Proposition 10.2.5. *1. μ is U_1-equivariant,*

2. γ is tangent to the fibres of μ,

3. μ is a conformal submersion from $\Sigma - \{0\}$ onto its image with conformal factor the square-norm in Σ.

Proof. 1. The Lie algebra of U_1 is 1-dimensional, so U_1-equivariance in this case is just U_1-invariance. Let $x \in \Sigma - \{0\}$ and choose $c_x : [0,1] \to \Sigma$ to be the straight line joining 0 to x. Then

$$c_{e^{i\theta}x} = e^{i\theta} c_x$$

and also note that $\gamma_{e^{i\theta}x} = (e^{i\theta})_* \gamma_x$ where $(e^{i\theta})_*$ is the derivative of the isometry $e^{i\theta}$. Each complex structure $J_{\mathfrak{j}}$ commutes with $(e^{i\theta})_*$. By Stokes' Theorem

$$\begin{aligned}
\mu_{\mathfrak{j}}(e^{i\theta}x) &= \int_{c_{e^{i\theta}x}} d\mu_{\mathfrak{j}} \\
&= \int_{c_{e^{i\theta}x}} J_{\mathfrak{j}}\gamma \\
&= \int_0^1 g\big(J_{\mathfrak{j}}\gamma(c_{e^{i\theta}x}(t)), c'_{e^{i\theta}x}(t)\big)dt \\
&= \int_0^1 g\big((e^{i\theta})_* J_{\mathfrak{j}}\gamma(c_x(t)), (e^{i\theta})_* c'_x(t)\big)dt \\
&= \int_0^1 g\big(J_{\mathfrak{j}}\gamma(c_x(t)), c'_x(t)\big)dt \\
&= \int_{c_x} J_{\mathfrak{j}}\gamma \\
&= \mu_{\mathfrak{j}}(x) \ .
\end{aligned}$$

2. This is trivial since from the formula for $\mu_{\mathfrak{i}}$, $\gamma(\mu_{\mathfrak{i}}) = g(J_{\mathfrak{i}}\gamma, \gamma) = 0$.

3. The derivative $\mu_* : T_x\Sigma \to T_{\mu(x)}V \overset{\text{canon}}{\cong} V$ is

$$\mu_* = (d\mu_1, \ldots, d\mu_7)$$

and hence

$$\mu_*(x) = (g(J_1\gamma, x), \ldots, g(J_7\gamma, x))$$

and we know that the fields $J_1\gamma, \ldots, J_7\gamma$ are orthogonal. Therefore $\mu_*(x)$ can be anything in V, i.e. μ_* is surjective, so μ is a submersion onto its image. The fields $J_1, \ldots, J_7$ span the horizontal distribution of μ, and since the $J_{\mathfrak{i}}$'s are isometries the basis elements $J_1, \ldots, J_7$ each have norm $||\gamma||$. If x is a horizontal vector tangent to Σ we can write its coordinates with $(x_1, \ldots, x_7)$ with respect to $J_1\gamma, \ldots, J_7\gamma$, and the square-norm of x is

$$||x||^2 = ||\gamma||^2 \sum_{\mathfrak{i}=1}^{7} x_{\mathfrak{i}}^2 \ .$$

The image of x is

$$\mu_*(x) = (x_1, \ldots, x_7) \in V$$

now written with respect to $\mathfrak{1}, \ldots, \mathfrak{7}$. This has square-norm $\sum_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{7}} x_{\mathfrak{i}}^2$, so μ is conformal with factor $||\gamma||^2$, and $||\gamma_x||^2 = ||x||^2$.

□

If we were working with an automorphism subgroup G of larger dimension than U_1 then we could ask if the above properties still hold. We have already mentioned the first of these in the context of hyper-Kähler moments, and similar remarks can be made more generally. In general, the vector fields generated by elements of the Lie algebra of G are not tangent to the fibres of a moment map. This is because a moment map takes values in $\mathfrak{g}^*$ (or each component does, if there are several complex structures) and a point in $\mathfrak{g}^*$ may not be fixed by the coadjoint action of G. Instead, we have to restrict our attention to the isotropy subgroup of a point in $\mathfrak{g}^*$, and those fields given by its Lie algebra will be tangent to a fibre. This is explained in [HKLR87]. Also, since the codomain of a moment map can have dimension larger than that of its domain, we cannot expect a moment map to be a submersion for other groups.
We denote the fibre of μ by

$$\mathcal{M}_\nu \stackrel{\text{def}}{=} \mu^{-1}(\nu) \ , \ \nu \in \mathrm{Im}\mu \subset V \ .$$

Note that all points in Σ are regular points of μ except 0, so $\mathcal{M}_\nu$ is always a manifold except when $\nu = 0$, in which case we can use the manifold $\mathcal{M}_0 - \{0\}$.

Proposition 10.2.6. *We have*

1. *$\mathcal{M}_\nu$ is preserved by the U_1-action,*
2. *$\mathcal{M}_\nu$ is complete,*
3. *$\mathcal{M}_\nu$ is spin.*

Proof. 1. This follows immediately from 2 of 10.2.5.

2. Let $x_1, x_2, \ldots$ be a Cauchy sequence in $\mathcal{M}_\nu$. This sequence is also Cauchy in the complete space Σ and therefore converges to a point $x \in \Sigma$. Since μ is smooth the sequence $\mu(x_1), \mu(x_2), \ldots$ converges to $\mu(x)$, but this is just the constant sequence $\nu, \nu \ldots$, so $\mu(x) = \nu$ and $x \in \mathcal{M}_\nu$.
3. The normal bundle $N\mathcal{M}_\nu$ is globally spanned by the orthogonal vector fields $J_{\mathfrak{1}}\gamma, \ldots, J_{\mathfrak{7}}\gamma$, so is trivial and oriented. The space Σ is parallelisable and oriented. By Proposition 2.15 of [LM89] a spin structure is uniquely specified on $\mathcal{M}_\nu$.

□

The manifold $\mathcal{M}_\nu$ certainly does not contain $0 \in \Sigma$, because $\mu(0) = 0$. This implies γ is nowhere vanishing on $\mathcal{M}_\nu$, and the action of U_1 on $\mathcal{M}_\nu$ is free. The action of the isometry group (and therefore of any subgroup) is always proper (see [Bes87]), so

$$\mathcal{B}_\nu \stackrel{\text{def}}{=} \mathcal{M}_\nu / \gamma$$

is a manifold. It is well-defined by 1 of Proposition 10.2.6 above. The U_1-action gives $\mathcal{M}_\nu$ the structure of a principal U_1-bundle over $\mathcal{B}_\nu$:

$$0 \to U_1 \to \mathcal{M}_\nu \to \mathcal{B}_\nu \to 0 \ .$$

A spin structure on the base and on each of the fibres of a submersion induces one on the total space. This doesn't work the other way around.

Definition 10.2.7. *([Amm99]) The U_1-action on $\mathcal{M}_\nu$ induces a U_1-action on the manifold $SO(\mathcal{M}_\nu)$. A spin structure on $\mathcal{M}_\nu$ is called* ***projectable*** *if this lifts to a U_1-action on $Spin(\mathcal{M}_\nu)$, i.e. if there is a map so that*

$$\begin{tikzcd}
& Spin(\mathcal{M}_\nu) \arrow[d] \\
& SO(\mathcal{M}_\nu) \arrow[d] \\
U_1 \arrow[r] \arrow[ur] \arrow[uur] & \mathcal{M}_\nu
\end{tikzcd}$$

commutes.

A projectable spin structure on $\mathcal{M}_\nu$ induces a spin structure on the quotient $\mathcal{B}_\nu$ (see [Amm98] or [Amm99]). It is not clear whether or not the spin structure we have on $\mathcal{M}_\nu$ is projectable, and so a different approach may be simpler.
For convenience we now adopt the notation

$$\Sigma_0 \overset{\text{def}}{=} \Sigma - \{0\} \ .$$

Proposition 10.2.8. *The diagram*

$$\begin{tikzcd}
\mathcal{M}_\nu \arrow[r, hook] \arrow[dd, "/U_1"] & \Sigma_0 \arrow[dd, "/U_1"] \arrow[dr, "\mu"] & \\
& & V \\
\mathcal{B}_\nu \arrow[r, hook] & \Sigma_0/U_1 \arrow[ur, "\mu^{/U_1}"'] &
\end{tikzcd}$$

commutes, where the vertical arrows are Riemannian submersions, the embeddings are isometric and $\mu^{/U_1}$ is the map induced by the moment map, and therefore also a conformal submersion.

Proof. The existence of $\mu^{/U_1}$ follows from the U_1-equivariance of μ proved in Proposition 10.2.5. The conformal factor $||\gamma||^2$ on Σ_0 is U_1-invariant and so lowers to Σ_0/U_1, which gives the conformal factor of $\mu^{/U_1}$. The space $\mathcal{B}_\nu$ is a level set of $\mu^{/U_1}$. □

Lemma 10.2.9. *The seven anticommuting complex structures on Σ induce seven anticommuting almost-complex structures $\check{J}_1, \dots, \check{J}_7$ on $\mathcal{B}_\nu$.*

Proof. Consider the restriction of $T\Sigma$ to $\mathcal{M}_\nu$:

$$T\Sigma|_{\mathcal{M}_\nu} = T\mathcal{M}_\nu \oplus N\mathcal{M}_\nu \ .$$

The normal bundle $N\mathcal{M}_\nu$ is spanned globally by $J_1\gamma, \dots, J_7\gamma$. Since the J_i's obey the octonionic multiplication rules of $1, \dots, 7$ (found in Appendix A), the bundle $\mathbb{R}\gamma \oplus N\mathcal{M}_\nu$ is invariant under each of the J_i's. The J_i's are orthogonal linear maps and therefore the orthogonal complement

$T\mathcal{M}_\nu^{\perp\gamma}$ is also invariant under each of the J_i's. The J_i's are U_1-invariant and so complex structures are induced on $\mathcal{B}_\nu$ as required. □

Lemma 10.2.10. *Denoting the Levi-Civita covariant derivative operators of $\mathcal{M}_\nu$ and $\mathcal{B}_\nu$ by ∇ and $\check{\nabla}$ respectively, we have*

$$\check{\nabla}_{\check{X}}\check{Y} = \nabla_X Y$$

where X, Y are the basic lifts of $\check{X}$, $\check{Y}$ by $\mathcal{M}_\nu \to \mathcal{B}_\nu$.

Proof. The group U_1 is clearly a closed and connected subgroup of the isometry group of $\mathcal{M}_\nu$, and the principal orbits of its action on $\mathcal{M}_\nu$ are all one-dimensional and therefore irreducible. The space of principal orbits is all of $\mathcal{M}_\nu$ because $\mathcal{M}_\nu$ does not contain $0 \in \Sigma$ by assumption. Therefore, Theorem 2.3.9 applies and $\mathcal{M}_\nu \to \mathcal{B}_\nu$ is a warped fibre bundle (as mentioned in Section 2.3). By Theorem 2.3.11, O'Neill's tensor field A vanishes. □

We can now prove the main result of this section.

Theorem 10.2.11. *The complex structures $\check{J}_1, \ldots, \check{J}_7$ are parallel and therefore $\mathcal{B}_\nu$ is flat.*

Proof. Let $\check{X}$ and $\check{Y}$ be vector fields on $\mathcal{B}_\nu$ and X and Y their basic lifts to $\mathcal{M}_\nu$. Denote the covariant derivative of Σ by ∇^Σ.

$$\begin{aligned}
\check{\nabla}_{\check{X}}(\check{J}_\mathrm{i}\check{Y}) &= \nabla_X(J_\mathrm{i}Y) \\
&= \left[\nabla^\Sigma_X(J_\mathrm{i}Y)\right]^{T\mathcal{M}_\nu^{\perp\gamma}} \\
&= \left[J_\mathrm{i}\nabla^\Sigma_X Y\right]^{T\mathcal{M}_\nu^{\perp\gamma}} \\
&= J_\mathrm{i}\left[\nabla^\Sigma_X Y\right]^{T\mathcal{M}_\nu^{\perp\gamma}} \\
&= J_\mathrm{i}\nabla_X Y \\
&= \check{J}_\mathrm{i}\check{\nabla}_{\check{X}}\check{Y} \ .
\end{aligned}$$

The holonomy group of $\mathcal{B}_\nu$ therefore consists of maps preserving seven anticommuting complex structures acting on an 8-dimensional vector space. In other words, it consists only of automorphisms of one of the irreducible real representations of the real Clifford algebra $C\ell_7$. But $C\ell_7$ is a sum of two copies of $\mathbb{R}(8)$ (non-canonically), and so the only such maps are multiplication by scalars. It follows that $\mathcal{B}_\nu$ is flat. □

This is a rather disappointing result, as we had hoped that $\mathcal{B}_\nu$ would have more interesting structure. Notice that by using only one of the complex structures, say J_1, the situation above is exactly that of the ordinary Kähler quotient of [HKLR87], mentioned in 10.1. This confirms that J_1 must induce a complex structure $\check{J}_1$, and this must be parallel.

10.3 Clifford structures

The existence of seven anticommuting parallel complex structures on the reduced space $\mathcal{B}_\nu$ of the 7-complex quotient proved to be too restrictive for $\mathcal{B}_\nu$ to be anything but flat. To suggest a possible alternative framework, we introduce the following notion due to Moroianu and Semmelmann. The reasons for this will become evident later.

Definition 10.3.1. *([MS10]) A rank r* ***Clifford structure***[1] *on a Riemannian manifold $(\mathcal{M}^n, g)$ is an oriented rank r Euclidean vector bundle (E, h) over $\mathcal{M}$ together with a non-trivial algebra bundle morphism $C\ell(E, h) \to \mathrm{End}T\mathcal{M}$ which maps $E \subset C\ell(E, h)$ into the bundle of skew-endomorphisms $\Lambda^2 T\mathcal{M}$.*

We may think of the bundle E as being a subbundle of $\Lambda^2 T\mathcal{M}$ such that unit local sections of E give us almost-complex structures on $\mathcal{M}$. A local orthonormal frame of E (with respect to h, not g because the morphism $C\ell(E, h) \to \mathrm{End}T\mathcal{M}$ isn't isometric) gives us r anticommuting almost-complex structures locally on $\mathcal{M}$.
As is remarked in [MS10], one may think of a Clifford structure as a dual notion to that of a spinor bundle (actually *pinor*—see Definition 10.3.4 below); the tangent bundle becomes the Clifford representation of the bundle $C\ell(E, h)$ of algebras associated to E.
There are several special cases of Clifford structures. A rank 1 Clifford structure is an almost-Hermitian structure; a rank 2 Clifford structure is an almost-hyper-Hermitian structure; a rank 3 Clifford structure is an almost-quaternionic-Hermitian structure. These examples are to be found in [MS10].

Remark 10.3.2. *A Clifford structure is a structure in the true sense, with structure group the normaliser of Pin_r in SO_n [MS10].*

A Clifford structure is called *flat* if the bundle E is trivialisable by parallel sections, and is called *parallel* if E is a parallel subbundle of $\Lambda^2 T\mathcal{M}$. In the three special cases mentioned above: a parallel rank 1 Clifford structure (which is automatically flat) is a Kähler structure; a parallel rank 2 Clifford structure is a hyper-Kähler structure; a parallel rank 3 Clifford structure is a quaternionic-Kähler structure.

Remark 10.3.3. *Notice that a rank 2 parallel Clifford structure is automatically flat since the curvature is forced to take values in $\mathfrak{sp}_n \oplus \mathfrak{u}_1 \subset \mathfrak{sp}_n \oplus \mathfrak{sp}_1$, and so must take values in the smaller space $\mathfrak{sp}_n \subset \mathfrak{sp}_n \oplus \mathfrak{u}_1$ by a theorem of Berger's (Corollary 14.47 in [Bes87]). The result then follows from the Ambrose-Singer Theorem.*

As Moroianu and Semmelmann point out, parallel Clifford structures can only exist either in low ranks $r \leq 3$ or in low dimensions $n \leq 8$, or on flat spaces. We will state this explicitly in Theorem 10.3.11 below. A less restrictive definition is more interesting.

Definition 10.3.4. *([MS10]) A rank r* ***even Clifford structure*** *on a Riemannian manifold $(\mathcal{M}^n, g)$ is an oriented rank r Euclidean vector bundle (E, h) over $\mathcal{M}$ together with a non-trivial algebra bundle morphism $C\ell^0(E, h) \to \mathrm{End}T\mathcal{M}$ which maps $\Lambda^2 E \subset C\ell^0(E, h)$ into the bundle of skew-endomorphisms $\Lambda^2 T\mathcal{M}$.*

Unlike for a Clifford structure, E can no longer be thought of as a subbundle of $\Lambda^2 T\mathcal{M}$, but $\Lambda^2 E$ can. Like a Clifford structure, we do get lots of local almost-complex structures on $\mathcal{M}$. If a local orthonormal frame of E is $e_1, \ldots, e_r$ then the image of $e_i \wedge e_j$ under $C\ell^0(E, h) \to \mathrm{End}T\mathcal{M}$ is an almost-complex structure for each pair i, j. However, these almost-complex structures do not all anticommute. Denoting the image of $e_i \wedge e_j$ in $\Lambda^2 T\mathcal{M}$ by J_{ij} (as is done in [MS10]) we have

[1] As pointed out in [MS10], the term 'Clifford structure' has been used by other authors in several different ways. The most common of these is the kind Moroianu and Semmelmann refer to as *flat*: when E is trivial so $T\mathcal{M}$ is a representation of a single algebra $C\ell(E, h)$ rather than a bundle of such algebras.

Lemma 10.3.5. *([MS10]) The local almost-complex structures J_{ij} satisfy*

$$J_{ij} \circ J_{ik} = J_{jk} \ , \quad J_{ij} \circ J_{kl} = J_{kl} \circ J_{ij}$$

for all i, j, k, l mutually distinct.

This is very easy to check. From this, we see that the local almost-complex structures J_{1i} form a maximal anticommuting set and there are r of these.
A Clifford structure always gives rise to an even Clifford structure. The images $J_1, \ldots, J_r$ of the basis members $e_1, \ldots, e_r$ under $C\ell(E,h) \to \mathrm{End}T\mathcal{M}$ allow us to write the image of $e_i \wedge e_j$ as

$$J_{ij} = J_i \circ J_j \ .$$

The converse does not always hold; the action of $C\ell^0(E,h)$ may not be extended to an action of $C\ell(E,h)$. If $r = 3 \mod 4$ then the bundle $C\ell(E,h)$ splits into a sum of two algebra bundles and there is a non-trivial homomorphism $C\ell(E,h) \to C\ell^0(E,h)$. The composition

$$C\ell(E,h) \to C\ell^0(E,h) \to \mathrm{End}T\mathcal{M}$$

is a non-trivial homomorphism, so defines a Clifford structure. There are other examples where an even Clifford structure can be extended to a Clifford structure. For example, the algebra $C\ell_6^0$ is non-canonically isomorphic to the real algebra of 4×4 complex matrices $\mathbb{C}(4)$ and therefore injects into $\mathbb{R}(8)$. Such an injective homomorphism always extends to a homomorphism $C\ell_6 \to \mathbb{R}(8)$, which is in fact an isomorphism. This tells us that an even Clifford structure of rank 6 on an 8-dimensional manifold can always be extended to a Clifford structure of rank 6. This case can be seen in the classification tables below. In [MS10] examples are also given of even Clifford structures which do not extend to Clifford structures.

Remark 10.3.6. *An even Clifford structure is a structure in the true sense, with structure group the normaliser of $Spin_r$ in SO_n [MS10].*

It is a little harder to define when an even Clifford structure is flat or parallel. Following [MS10], we say an even Clifford structure is *parallel* if there is a Euclidean covariant derivative on E such that the induced covariant derivative on $\Lambda^2 E$ agrees with the covariant derivative of $\Lambda^2 T\mathcal{M}$. Further, an even Clifford structure is *flat* if the bundle E is flat.
Similarly to the Clifford structure case, there are special cases of even Clifford structures which we are already familiar with. Notably, a rank 2 even Clifford structure E is an almost-Hermitian structure that is Kähler if and only if E is parallel; a rank 3 even Clifford structure E is an almost-quaternionic-Hermitian structure that is quaternionic-Kähler if and only if E is parallel. These examples are again to be found in [MS10].

Remark 10.3.7. *As can be seen from the examples we have given, the difference between flat (even) Clifford structures and parallel ones is analogous to the difference between hyper-Kähler and quaternionic-Kähler manifolds.*

Lemma 10.3.8. *If a Clifford structure splits into parallel subbundles $E = E_1 \oplus E_2$ (i.e. it is holonomy-reducible) then E_1 and E_2 are Clifford structures in a natural way.*

Proof. The non-trivial homomorphism $C\ell(E) \to \mathrm{End}T\mathcal{M}$ gives us homomorphisms $C\ell(E_1) \to$

$\mathrm{End}T\mathcal{M}$ and $C\ell(E_2) \to \mathrm{End}T\mathcal{M}$ using the natural isomorphism and inclusions

$$C\ell(E_1), C\ell(E_2) \subset C\ell(E_1) \hat{\otimes} C\ell(E_2) \overset{\text{canon}}{\cong} C\ell(E) .$$

Any real Clifford algebra is either a matrix algebra or a sum of two matrix algebras. In the first case, any non-zero map $C\ell(E) \to \mathrm{End}T\mathcal{M}$ is injective, so must be injective on E. In the second case, E is diagonal in the sum and it is easy to see its intersection with the kernel is trivial. The non-trivial homomorphism $C\ell(E) \to \mathrm{End}T\mathcal{M}$ therefore always restricts to an injective map $E \to \Lambda^2 T\mathcal{M}$, so we get injective maps $E_1, E_2 \to \Lambda^2 T\mathcal{M}$. Injectivity implies non-triviality, so it follows E_1 and E_2 are Clifford structures. □

Remark 10.3.9. *The observation*

$$C\ell^0(E_1) \hat{\otimes} C\ell^0(E_2) + C\ell^1(E_1) \hat{\otimes} C\ell^1(E_2) \overset{\text{canon}}{\cong} C\ell^0(E)$$

allows us to define homomorphisms $C\ell^0(E_1), C\ell^0(E_2) \to \mathrm{End}T\mathcal{M}$*. The non-trivial homomorphism* $C\ell^0(E) \to \mathrm{End}T\mathcal{M}$ *restricts to an injective map* $\Lambda^2 E \to \Lambda^2 T\mathcal{M}$ *and therefore the same proof also works to prove Lemma 10.3.8 for even Clifford structures.*

Even Clifford structures offer greater generality than ordinary Clifford structures. This is evident in the following two theorems, which we present in the order they appear in [MS10]. Moroianu and Semmelmann fully classify both Clifford structures and even Clifford structures in the flat and parallel cases over complete simply connected Riemannian manifolds.

Theorem 10.3.10. *([MS10]) The list of complete simply connected Riemannian manifolds* $\mathcal{M}$ *carrying a parallel rank* r even *Clifford structure is given in the tables below.*

Manifolds with a flat *even Clifford structure:*

r	$\mathcal{M}$	$\dim_{\mathbb{R}} \mathcal{M}$
2	*Kähler*	$2m,\ m \geq 1$
3	*hyper-Kähler*	$4m,\ m \geq 1$
4	*reducible hyper-Kähler*	$4(m_1 + m_2),\ m_1 \geq 1,\ m_2 \geq 0$
arbitrary	$C\ell^0_r$ *representation space*	

Manifolds with a parallel non-flat *even Clifford structure:*

r	$\mathcal{M}$	$\dim_{\mathbb{R}} \mathcal{M}$
2	*Kähler*	$2m,\ m \geq 1$
3	*quaternionic-Kähler*	$4m,\ m \geq 1$
5	*quaternionic-Kähler*	8
6	*Kähler, spin*	8
7	$Spin_7$	8
8	*generic holonomy, spin*	8
5	$Sp_{k+2}/Sp_k \times Sp_2$	$8k,\ k \geq 2$
8	$SO_{k+8}/SO_k \times SO_8$	$8k,\ k \geq 2$ *even*
9	$\mathbb{O}\mathbb{P}^2 = F_4/Spin_9$	16
10	$(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = E_6/Spin_{10} \cdot U_1$	32
12	$(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2 = E_7/Spin_{12} \cdot SU_2$	64
16	$(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2 = E_8/Spin_{16}$	128
5	$Sp_{k,2}/Sp_k \times Sp_2$	$8k,\ k \geq 2$
8	$SO^0_{k,8}/SO_k \times SO_8$	$8k,\ k \geq 2$ *even*
9	$\mathbb{O}H^2 = F_4^{-20}/Spin_9$	16
10	$(\mathbb{C} \otimes \mathbb{O})H^2 = E_6^{-14}/Spin_{10} \cdot U_1$	32
12	$(\mathbb{H} \otimes \mathbb{O})H^2 = E_7^{-5}/Spin_{12} \cdot SU_2$	64
16	$(\mathbb{O} \otimes \mathbb{O})H^2 = E_8^8/Spin_{16}$	128

The tables in [MS10] are presented slightly differently. The non-compact duals in the final six rows are not included, and those authors also consider a different structure called a *projective even Clifford structure*. We will not be interested in these.

Theorem 10.3.11. *([MS10]) A complete simply connected Riemannian manifold $\mathcal{M}^n$ carries a parallel rank r Clifford structure if and only if one of the following (non-exclusive) cases occurs:*

1. *$r = 1$ and $\mathcal{M}$ is Kähler*

2. *$r = 2$ and either $n = 4$ and $\mathcal{M}$ is Kähler or $n \geq 8$ and $\mathcal{M}$ is hyper-Kähler*

3. *$r = 3$ and $\mathcal{M}$ is quaternionic-Kähler*

4. *$r = 4$, $n = 8$ and $\mathcal{M}$ is a product of two Calabi-Yau 4-manifolds*

5. *$r = 5$, $n = 8$ and $\mathcal{M}$ is hyper-Kähler*

6. *$r = 6$, $n = 8$ and $\mathcal{M}$ is Calabi-Yau*

7. *$r = 7$, $n = 8$ and $\mathcal{M}$ is $Spin_7$*

8. *r is arbitrary and $\mathcal{M}$ is flat, isometric to a non-trivial representation of $C\ell_r$*

The proofs of these two theorems are long so we will not reproduce them here. Consider the space $\mathcal{B}_\nu$ as described in Section 10.2. Lemma 10.2.9 and Theorem 10.2.11 show that $\mathcal{B}_\nu$ naturally inherits seven anticommuting parallel complex structures. In other words, $\mathcal{B}_\nu$ inherits a flat Clifford structure of rank 7. Theorem 10.3.11 then confirms our result that $\mathcal{B}_\nu$ indeed must be flat. It also tells us what structure we would really like $\mathcal{B}_\nu$ to have: a parallel non-flat Clifford structure of rank 7. Note that since $7 = 3$ modulo 4, this is equivalent to a parallel non-flat even Clifford structure of rank 7.
The space Σ we used to construct $\mathcal{B}_\nu$ in Section 10.2 also appears in the list in Theorem 10.3.11; Σ is a flat representation space of $C\ell_7$. So Σ has a flat Clifford structure of rank 7 as well. The new idea is to choose a different space, that we shall also call Σ, to begin with. If Σ possesses a different kind of Clifford structure then perhaps $\mathcal{B}_\nu$ will inherit a parallel non-flat Clifford structure of rank 7 as we want, by a similar procedure. Note that Σ cannot be a $Spin_7$-manifold because the quotient must reduce the dimension (see Appendix E for interesting facts about actions on $Spin_7$-manifolds), so we must choose it to be one of the bigger spaces from the above table. The 7-complex quotient of Section 10.2 is analogous to the hyper-Kähler quotient of [HKLR87], and our new type of 'Clifford quotient' will be the analogue of the quaternionic-Kähler quotient of Galicki and Lawson [GL88], which we now briefly describe.

10.4 Clifford quotients

Let (Σ^{4n}, g) be locally quaternionic-Kähler, i.e. a Riemannian manifold with restricted holonomy contained in $Sp_n \cdot Sp_1$. Denote by E the three-dimensional parallel subbundle of the skew-endomorphism bundle $\Lambda^2 T\Sigma$. If J_1, J_2, J_3 is a local frame of E consisting of anticommuting almost-complex structures, the *fundamental* 4*-form* of (Σ, g) is defined to be

$$\Omega \stackrel{\text{def}}{=} J_1 \wedge J_1 + J_2 \wedge J_2 + J_3 \wedge J_3 \ .$$

Obviously Ω is independent of the choice of local frame J_1, J_2, J_3 and is parallel. In fact Ω fully characterises the quaternionic-Kähler structure; the stabiliser of Ω is $Sp_n \cdot Sp_1$. The group of isometries of (Σ, g) (which automatically preserve E) will be denoted $\mathrm{Aut}(\Sigma)$. Write

$$\Theta \stackrel{\text{def}}{=} J_1 \otimes J_1 + J_2 \otimes J_2 + J_3 \otimes J_3$$

so that Θ is an E-valued section of E that antisymmetrises to Ω. Again, Θ is independent of the choice of local frame J_1, J_2, J_3.

We first need to understand the curvature of the bundle E. Denote by ∇^E the covariant derivative operator on E, and $\nabla^{E^\perp}$ that on the orthogonal complement $E^\perp$. Since E is a parallel subbundle of $\Lambda^2 T\Sigma$ we can write

$$\nabla^{\Lambda^2 T\Sigma} = \begin{pmatrix} \nabla^E & 0 \\ 0 & \nabla^{E^\perp} \end{pmatrix}$$

with respect to $\Lambda^2 T\Sigma = E \oplus E^\perp$. If R^E, $R^{E^\perp}$ and $R^{\Lambda^2 T\Sigma}$ are the corresponding curvature tensors, it is easy to show that

$$R^{\Lambda^2 T\Sigma} = \begin{pmatrix} R^E & 0 \\ 0 & R^{E^\perp} \end{pmatrix} .$$

We shall use the following lemma, which we prove in the setting of general Clifford structures.

Lemma 10.4.1. *For a Clifford structure E of rank $r \geq 2$ over Σ^m ($m > 4$), the curvature tensor R^E can be identified with a section of $\Lambda^3 E$.*

Proof. It is clear that R^E is a section of $\Lambda^2 T\Sigma \otimes \Lambda^2 E$. If we write

$$\nabla^E J_i = \sum_{j=1}^{r} \alpha_{ij} \otimes J_j$$

for local 1-forms α_{ij}, then $\alpha_{ij} = (1/m) g(\nabla^E J_i, J_j)$ and it follows that $\alpha_{ji} = -\alpha_{ij}$. The α_{ij}'s are the coefficients of ∇^E with respect to the local g-orthonormal frame $(1/\sqrt{m})J_1, \ldots, (1/\sqrt{m})J_r$ of E. A calculation using vector fields X and Y then gives

$$\begin{aligned} R^E_{X\wedge Y}(J_i) &= \nabla^E_X \nabla^E_Y J_i - \nabla^E_Y \nabla^E_X J_i - \nabla^E_{[X,Y]} J_i \\ &= 2\sum_{j=1}^{r} d\alpha_{ij}(X\wedge Y) J_j - \sum_{k=1}^{r} \alpha_{ij} \wedge \alpha_{jk}(X \wedge Y) J_k \ . \end{aligned}$$

Therefore

$$g(R^E(J_i), J_j) = 2m\big[d\alpha_{ij} - \sum_{k=1}^{r} \alpha_{ik} \wedge \alpha_{kj}\big] \ .$$

Using the fact that $\nabla^{\Lambda^2 T\Sigma} J_i = \nabla^E J_i$ it is clear that $dJ_i = \sum_{j=1}^r \alpha_{ij} \wedge J_j$. Thus

$$0 = 2d^2 J_i = 2\sum_{j=1}^{r} d\alpha_{ij} \wedge J_j - \alpha_{ij} \wedge \sum_{k=1}^{r} \alpha_{jk} \wedge J_k \ .$$

This is equivalent to

$$\sum_{j=1}^{r} g(R^E(J_i), J_j) \wedge J_j = 0 \ .$$

Lemma B.0.26 then tells us $g(R^E(J_i), J_j)$ is a section of E for every pair i, j, and moreover

$$g(R^E(J_i), J_j) = \sum_{k=1}^{r} \lambda_{ijk} J_k \ , \quad \lambda_{ikj} = -\lambda_{ijk} \ .$$

□

Lemma 10.4.1 is more general than is required for the quaternionic-Kähler case because in that case, $\Lambda^3 E$ is a line bundle. Choose anticommuting local almost-complex structures J_1, J_2, J_3 on a quaternionic-Kähler manifold and define the 2-forms η_{12}, η_{13}, η_{21}, η_{23}, η_{31}, η_{32} by

$$\begin{aligned}
[R_{X\wedge Y}, J_1] &= \phantom{\eta_{21}(X\wedge Y)J_1 + {}} \eta_{12}(X\wedge Y)J_2 + \eta_{13}(X\wedge Y)J_3 \ , \\
[R_{X\wedge Y}, J_2] &= \eta_{21}(X\wedge Y)J_1 \phantom{{}+\eta_{12}(X\wedge Y)J_2} + \eta_{23}(X\wedge Y)J_3 \ , \\
[R_{X\wedge Y}, J_3] &= \eta_{31}(X\wedge Y)J_1 + \eta_{32}(X\wedge Y)J_2 \phantom{{}+\eta_{23}(X\wedge Y)J_3} \ .
\end{aligned}$$

It is easy to show that

$$R^E_{X\wedge Y}(J_i) = [R_{X\wedge Y}, J_i]$$

and so we can see that

$$\eta_{ij} = 4ng(R^E(J_i), J_j) \ .$$

The following is a lemma of Berger.

Lemma 10.4.2. *([Bes87]) On a quaternionic-Kähler manifold of real dimension $4n$, the function $\eta_{ij}(X \wedge J_iJ_jY)$ is independent of i,j and moreover*

$$\eta_{ij}(X\wedge J_iJ_jY) = \frac{1}{n+2} Ric(X \vee Y)$$

where $\vee$ is symmetric wedge.

The following theorem is contained in [GL88], but is not presented as a theorem there.

Theorem 10.4.3. *Let (Σ^{4n}, g) be a quaternionic-Kähler manifold and denote by E its Clifford structure. Then*

$$R^E = \frac{3\kappa}{8n^{\frac{5}{2}}(n+2)} vol_E$$

where κ is the scalar curvature and vol_E is the volume form on E with respect to g.

Proof. Locally, we can always choose an orthonormal frame

$$e_1, \dots, e_n, J_1e_1, \dots, J_1e_n, J_2e_1, \dots, J_2e_n, J_3e_1, \dots, J_3e_n$$

such that the J_i's act by permutations on it (such a frame is used in the proof of Lemma 10.4.2). Using this frame we can write

$$\begin{aligned}
J_1 &= 2\sum_{i=1}^n e_i \wedge J_1e_i + J_2e_i \wedge J_3e_i \ , \\
J_2 &= 2\sum_{i=1}^n e_i \wedge J_2e_i + J_3e_i \wedge J_1e_i \ , \\
J_3 &= 2\sum_{i=1}^n e_i \wedge J_3e_i + J_1e_i \wedge J_2e_i \ .
\end{aligned}$$

We can check that when J_i acts on e_j it gives J_ie_j, and that the quaternionic algebra rules are satisfied by these expressions. Using Lemma 10.4.2,

$$\begin{aligned}
4ng(R^E_{J_3}(J_1), J_2) &= 4n\sum_{i=1}^{n} g(R^E_{e_i\wedge J_3 e_i}(J_1), J_2) + g(R^E_{J_1 e_i\wedge J_3 J_1 e_i}(J_1), J_2)\\
&= \sum_{i=1}^{n} \eta_{12}(e_i\wedge J_3 e_i) + \eta_{12}(J_1 e_i \wedge J_3 J_1 e_i)\\
&= \sum_{i=1}^{n} \eta_{12}(e_i\wedge J_1 J_2 e_i) + \eta_{12}(J_1 e_i \wedge J_1 J_2 J_1 e_i)\\
&= \frac{1}{n+2}\sum_{i=1}^{n} Ric(e_i\vee e_i) + Ric(J_1 e_i\vee J_1 e_i)\\
&= \frac{2}{n+2}\sum_{i=1}^{n} Ric(e_i\vee e_i)\\
&= \frac{2}{n+2}\kappa\ .
\end{aligned}$$

We have used the fact that the manifold is Einstein to show $Ric(J_1 e_i \vee J_1 e_i) = Ric(e_i \vee e_i)$. $\square$

Theorem 10.4.4. *([GL88]) Let (Σ^{4n}, g) be a quaternionic-Kähler manifold with $4n \geq 8$ and let E denote its Clifford structure. If the scalar curvature κ of Σ is non-zero, there exists a unique section μ of $\mathfrak{aut}(\Sigma)^* \otimes E$ on Σ such that*

$$\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$$

for every quaternionic-Kähler Killing vector field $\gamma \in \mathfrak{aut}(\Sigma)$. Furthermore, μ is given by the expression

$$\mu(\gamma) = \frac{16n^{\frac{9}{2}}(n+2)}{\kappa} *_E (\mathscr{L}_\gamma - \nabla^E_\gamma)$$

where κ is the scalar curvature of Σ and $*_E : \Lambda^2 E \to E$ is the Hodge star operator of the three-dimensional bundle E with respect to g. The precise meaning of the formula for $\mu(\gamma)$ is explained at the end of the proof.

Proof. We will loosely follow the proof of Galicki and Lawson, but our proof will be written differently so that we may bear in mind a method of proving a similar theorem for more general Clifford structures. Note that again we choose to use the same notation for the element $\gamma \in \mathfrak{aut}(\Sigma)$ and the vector field on Σ it generates.

Applying the exterior covariant derivative operator d^{∇^E} to both sides of the equation $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ gives

$$R^E\big[\mu(\gamma)\big] = d^{\nabla^E}\big[\Theta(\gamma)\big]\ .$$

If we can find $\mu(\gamma)$ satisfying this equation, it will also satisfy $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ up to a parallel section of E, of which there are none (see Lemma 10.4.5 below). In terms of the local basis J_1, J_2, J_3 of E, we may write

$$\mu(\gamma) = \sum_{i=1}^{3} \mu_i(\gamma) J_i\ , \quad \Theta(\gamma) = \sum_{i=1}^{3} J_i\gamma \otimes J_i\ .$$

Lemma 10.4.1 shows that we can write $R^E(J_i) = (1/4n)\sum_{j,k=1}^{3}\lambda_{ijk}J_j \wedge J_k$ and therefore

$$R^E\big[\mu(\gamma)\big] = \frac{1}{4n}\sum_{i,j,k=1}^{3}\mu_i(\gamma)\lambda_{ijk}J_j \wedge J_k \ .$$

Also

$$d^{\nabla^E}\big[\Theta(\gamma)\big] = \sum_{i=1}^{3}\Big[d(J_i\gamma) + \sum_{j=1}^{3}\alpha_{ji}\wedge J_j\gamma\Big]\otimes J_i \ .$$

Whence, there exists a section $\mu(\gamma)$ satisfying $R^E\big[\mu(\gamma)\big] = d^{\nabla^E}\big[\Theta(\gamma)\big]$ if and only if

$$d(J_i\gamma) + \sum_{j=1}^{3}\alpha_{ji}\wedge J_j\gamma = \sum_{k=1}^{3}\beta_{ik}J_k$$

for functions β_{ik} with $\beta_{ki} = -\beta_{ik}$. The condition $\mathscr{L}_\gamma\Omega = 0$ implies

$$0 = \sum_{i=1}^{3}\Big[d(J_i\gamma) + \sum_{j=1}^{3}\alpha_{ji}\wedge J_j\gamma\Big]\wedge J_i \ .$$

Lemma B.0.26 then gives the existence of the β_{ik}'s. This shows that μ exists. To simplify things we could have used the formula of Theorem 10.4.3 but we chose not to in order to keep in mind the general situation. Notice as well that there are two kinds of wedge product in action here; that of $T\Sigma$ and that of E. Since we've shown that $d(J_i\gamma) + \sum_{j=1}^{3}\alpha_{ji}\wedge J_j\gamma$ is a linear combination of the J_k's with skew coefficients, we could write

$$d^{\nabla^E}\big[\Theta(\gamma)\big] = \sum_{i=1}^{3}\Big[d(J_i\gamma) + \sum_{j=1}^{3}\alpha_{ji}\wedge J_j\gamma\Big]\wedge J_i \ .$$

However, in this expression, unlike that resulting from $\mathscr{L}_\gamma\Omega = 0$, the final wedge product is that belonging to E and not to $T\Sigma$. This expression does not vanish—this would be impossible for $d^{\nabla^E}\big[\Theta(\gamma)\big]$ if it is to be equal to $R^E\big[\mu(\gamma)\big]$. This is the only confusion of wedges, so we just use the same notation for both and remain careful.

If μ' is another section of $\mathfrak{aut}(\Sigma)^*\otimes E$ satisfying $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ then the difference $\mu(\gamma)-\mu'(\gamma)$ is a parallel section of E. However,

Lemma 10.4.5. *The Clifford structure E of a quaternionic-Kähler manifold admits a non-zero parallel section if and only if the manifold is hyper-Kähler.*

Proof. It is obvious that E admits a section if the manifold is hyper-Kähler. The converse direction follows from Lemma 10.3.8 and Remark 10.3.3. ☐

This proves that μ is unique. The formula for μ is found in [GL88] as follows.

$$\begin{aligned}\mathscr{L}_\gamma J_i &= 3dJ_i(\gamma) + 2d(J_i\gamma)\\ &= 3\sum_{j=1}^{3}(\alpha_{ij}\wedge J_j)(\gamma) + 2d(J_i\gamma)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{3} \big[\alpha_{ij}(\gamma)J_j - 2\alpha_{ij} \wedge J_j\gamma\big] + 2d(J_i\gamma) \\
&= \nabla^E_\gamma J_i + 2\sum_{j=1}^{3} \alpha_{ji} \wedge J_j\gamma + 2d(J_i\gamma) \\
&= \nabla^E_\gamma J_i + \frac{2}{4n} d^{\nabla^E}\big[\Theta(\gamma)\big](J_i) \\
&= \nabla^E_\gamma J_i + \frac{2}{4n} R^E\big[\mu(\gamma)\big](J_i).
\end{aligned}
$$

Using the formula of Theorem 10.4.3 this gives

$$
\frac{\kappa}{8n^{\frac{5}{2}}(n+2)} *_E \mu(\gamma)(J_i) = 2n\big[\mathscr{L}_\gamma J_i - \nabla^E_\gamma J_i\big]
$$

from which the formula follows, noting that $*_E^2 = 1$. In the form it is written, $\mathscr{L}_\gamma$ and ∇^E_γ act on sections of E and their difference $\mathscr{L}_\gamma - \nabla^E_\gamma$ is a skew endomorphism of E, i.e. a section of $\Lambda^2 E$. The Hodge star then gives us a section of E. □

Consider the action of a group $G < \mathrm{Aut}(\Sigma)$ on Σ.

Definition 10.4.6. *The **quaternionic-Kähler moment map** of a G-action is the section μ^G of $\mathfrak{g}^* \otimes E$ given by the restriction of μ in Theorem 10.4.4 to $\mathfrak{g} \subset \mathfrak{aut}(\Sigma)$. We denote its zero set by Z_{μ^G}.*

Notice the similarity between this moment map and the hyper-Kähler version, which is the special case when ∇^E is a flat connection. In [GL88] it is explained that the hyper-Kähler moment map is obtained by integration of its defining equation and this leads to a constant of integration, whereas the quaternionic-Kähler moment map is obtained by differentiation of its defining equation. This means that the quaternionic-Kähler version is better behaved—it always exists uniquely. The map μ is $\mathrm{Aut}(\Sigma)$-equivariant so the quaternionic-Kähler moment map is G-equivariant, and this can easily be observed from the invariance of the equation $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ along with the uniqueness of its solution.
The hyper-Kähler quotient construction allows us to choose any regular value of the moment map and consider its preimage. The Clifford structure of a quaternionic-Kähler manifold with non-zero scalar curvature is non-trivial and so the only level set that makes sense is the zero set Z_{μ^G}, as noted in [GL88].

Theorem 10.4.7. *([GL88]) Let Σ be a quaternionic-Kähler manifold whose scalar curvature κ is non-zero, and let $G < \mathrm{Aut}(\Sigma)$ be a compact subgroup with moment map μ^G. The unique Riemannian metric on Z_{μ^G}/G which makes $Z_{\mu^G} \to Z_{\mu^G}/G$ a Riemannian submersion is quaternionic-Kähler.*

The zero set Z_{μ^G} is automatically G-invariant. In general Z_{μ^G}/G is an orbifold, and [GL88] contains conditions for it to be a manifold—instead of using Z_{μ^G} we use the largest G-invariant subset of the set of points at which μ^G intersects the zero section transversally and on which G acts freely. To avoid these technicalities we could restrict our attention to the parts of Z_{μ^G} that do form a manifold, or we could just accept it as an orbifold. We do not need to know how to prove this theorem and we present it mainly to motivate the study of moment maps for actions on spaces with Clifford structures, though the proof is not so difficult and actually uses an O'Neill formula for a Riemannian submersion.

Galicki and Lawson remark in [GL88] that one can use the expression for R^E in Theorem 10.4.3 to prove the existence and uniqueness parts of Theorem 10.4.4 more easily, and that they do not prove it that way in order to illustrate the point more carefully to those who may consider other generalisations of the quotient construction. Since we are doing exactly that, their forethought has proven very useful. To get the explicit formula for $\mu(\gamma)$ we do need Theorem 10.4.3, but for existence and uniqueness we need only show that R^E, considered as a map $E \to \Lambda^2 E$, is injective and that $d^{\nabla^E}\left[\Theta(\gamma)\right]$ lies in its image. This method will work for other Clifford structures, when we do not have a nice expression for the 3-form R^E. We will also need a replacement for Lemma 10.4.5.

Theorem 10.4.8. *The parallel even Clifford structures on the following spaces admit no non-trivial parallel subbundles, and in particular no non-zero parallel sections.*

1. $SO_{k+8}/(SO_k \times SO_8)$; $r = 8$, k *even,*
2. $\mathbb{OP}^2$; $r = 9$,
3. $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$; $r = 10$,
4. $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$; $r = 12$,
5. $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$; $r = 16$.

Proof. This follows from the classification table of Moroianu and Semmelmann in Theorem 10.3.10 and Lemma 10.3.8. □

Since we decided that the big space Σ must have dimension greater than eight and must have a Clifford structure or even Clifford structure of rank at least seven, only the above spaces remain as candidates. It should come as no surprise that in constructing a Riemannian manifold with exceptional holonomy we have to use one of a finite number of exceptional phase spaces, rather than one of an infinite family of spaces.

If $\mathcal{M}$ is a Riemannian manifold with $Spin_7$ holonomy and $B_{Spin_7^+}$ is one of its holonomy subbundles, we have the parallel splitting

$$\Lambda^2 T\mathcal{M} = B_{Spin_7^+} \times_{Spin_7^+} \mathfrak{spin}_7^+ \oplus \Psi_0^\perp$$

into subbundles associated to the representations $\mathfrak{spin}_7^+$ and $\Psi_0^\perp$. It is easy to check that the second summand is a parallel Clifford structure of rank 7 and the first summand is a parallel even Clifford structure of rank 7. These admit parallel subbundles if and only if the holonomy is contained in the proper subgroup $SU_4^+ < Spin_7^+$. In this case the second exterior power splits further into pieces associated to the SU_4^+-irreducible splittings $\mathfrak{spin}_7^+ = \mathfrak{su}_4^+ \oplus C^6$ and $\Psi_0^\perp = \mathbb{R}\chi_1 \oplus C^6$. Here C^6 gives us a parallel Clifford structure of rank 6 whose corresponding parallel even Clifford structure of rank 6 is the bundle associated to $\mathfrak{su}_4^+$. The section χ_1 of $\Psi^\perp$ is the second parallel spinor field of the same chirality as Ψ present on a Calabi-Yau manifold of dimension eight, as described by Wang's Theorem 1.0.1. This Clifford structure admits a parallel subbundle if and only if the holonomy is contained in the proper subgroup $Sp_2^+ < SU_4^+$. In this case the second exterior power splits further into pieces associated to the Sp_2^+-irreducible splittings $\mathfrak{su}_4^+ = \mathfrak{sp}_2^+ \oplus C^5$ and $C^6 = \mathbb{R}\chi_2 \oplus C^5$. Here C^5 gives us a parallel Clifford structure of rank 5 whose corresponding parallel even Clifford structure of rank 5 is the bundle associated to $\mathfrak{sp}_2^+$. The section χ_2 of $\Psi^\perp$ is the third parallel spinor field of the same chirality as Ψ present on

a hyper-Kähler manifold of dimension eight. If the holonomy is contained in $Sp_1 \times Sp_1 < Sp_2$ we get further splittings and a rank 4 Clifford structure, and finally if the holonomy is trivial the Clifford structure is forced to be flat. In this way we have worked our way up the list of even Clifford structures and Clifford structures of Theorems 10.3.10 and 10.3.11 in terms of the holonomy group. We would like to find a space $\mathcal{M}$ with holonomy exactly $Spin_7$, which means we do not want the parallel rank 7 Clifford structure to have any parallel subbundles.
If we are going to find a Clifford analogue of Theorem 10.4.4 we need a version of the fundamental 4-form Ω defined on a quaternionic-Kähler manifold, because this is needed for the proof.

Definition 10.4.9. *Let Σ^m be a Riemannian manifold with a Clifford structure E of rank r and let $J_1, \dots, J_r$ be a local frame of E consisting of anticommuting almost-complex structures. The **Clifford 4-form** is defined by*

$$\Omega \stackrel{\text{def}}{=} \sum_{i=1}^{r} J_i \wedge J_i \ .$$

Note that Ω is independent of our choice of frame $J_1, \dots, J_r$ and so is a canonical globally-defined 4-form on Σ.

It is not clear if Ω fully characterises the Clifford structure. We know a Clifford structure may be considered as a structure in the true sense, whose structure group is the normaliser of Pin_r in SO_m (see [MS10]). It would be an interesting exercise to study the splittings of the exterior powers $\Lambda^i\mathbb{R}^m$ into irreducibles with respect to this subgroup, as we have exhibited for $Spin_7$ in Proposition 8.1.9. This may enable us to classify Clifford structures using the procedure of Gray and Hervella.

Proposition 10.4.10. *If the Clifford structure is parallel then Ω is parallel.*

Proof. This is easy. □

Proposition 10.4.11. *If Σ has the structure listed below then the Clifford 4-form Ω is as follows:*

1. *Σ is Kähler: Ω is the square of the Kähler form,*
2. *Σ is quaternionic-Kähler: Ω is the fundamental 4-form,*
3. *Σ is $Spin_7$: Ω is $-4!$ times the Cayley form Φ.*

Proof. Statements *1* and *2* are definitions. For *3*, we need a local frame of the Clifford structure $\Psi^\perp$. Such a frame was given in terms of a Cayley frame in Section 8.1, and we define

$$\begin{aligned}
J_1 &= 2(0 \wedge 1 - 2 \wedge 3 - 4 \wedge 5 + 6 \wedge 7) \ , \\
J_2 &= 2(0 \wedge 2 + 1 \wedge 3 - 4 \wedge 6 - 5 \wedge 7) \ , \\
J_3 &= 2(0 \wedge 3 - 1 \wedge 2 - 4 \wedge 7 + 5 \wedge 6) \ , \\
J_4 &= 2(0 \wedge 4 + 1 \wedge 5 + 2 \wedge 6 + 3 \wedge 7) \ , \\
J_5 &= 2(0 \wedge 5 - 1 \wedge 4 + 2 \wedge 7 - 3 \wedge 6) \ , \\
J_6 &= 2(0 \wedge 6 - 1 \wedge 7 - 2 \wedge 4 + 3 \wedge 5) \ , \\
J_7 &= 2(0 \wedge 7 + 1 \wedge 6 - 2 \wedge 5 - 3 \wedge 4) \ .
\end{aligned}$$

One can check that these are anticommuting almost-complex structures all lying in $\Psi^{\perp}$. A simple calculation gives the squares, for example

$$J_1 \wedge J_1 = -8(0 \wedge 1 \wedge 2 \wedge 3 + 0 \wedge 1 \wedge 4 \wedge 5 - 0 \wedge 1 \wedge 6 \wedge 7 \\ -2 \wedge 3 \wedge 4 \wedge 5 + 2 \wedge 3 \wedge 6 \wedge 7 + 4 \wedge 5 \wedge 6 \wedge 7) \ .$$

It follows that $\Omega = \sum_{i=1}^{7} J_i \wedge J_i = -4!\Phi$. $\square$

The structures considered in Proposition 10.4.11 are parallel, but the same result holds for non-parallel structures: almost-Hermitian, almost-quaternionic-Hermitian and $Spin_7$-structures.

The Clifford 4-form Ω may not seem like a new object, given the few possible parallel Clifford structures in Theorem 10.3.11. However, we do not know if there are unfamiliar Clifford structures that are not parallel for which Ω will be new. It is also interesting to see that the 4-forms of Proposition 10.4.11 can be viewed as examples of the same object in this way.

For a choice of anticommuting local almost-complex structures $J_1, \ldots, J_r$ we have the canonical element

$$\Theta \stackrel{\text{def}}{=} \sum_{i=1}^{r} J_i \otimes J_i \ .$$

Note that Θ is just $m = \dim \Sigma$ times the identity on E, and that Θ antisymmetrises to Ω. The main result of this section is the following generalisation of Theorem 10.4.4 to Clifford structures.

Theorem 10.4.12. *Let $(\Sigma^{m>4}, g)$ be a Riemannian manifold with parallel Clifford structure E of rank $r \geq 2$ that has no non-zero parallel sections. Then there exists a unique section μ of $\mathfrak{aut}(\Sigma)^* \otimes E$ on Σ such that*

$$\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$$

for every Clifford-Killing vector field $\gamma \in \mathfrak{aut}(\Sigma)$.

Proof. The proof is very similar to that of Theorem 10.4.4. Applying the exterior covariant derivative operator d^{∇^E} to both sides of the equation $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ gives

$$R^E\big[\mu(\gamma)\big] = d^{\nabla^E}\big[\Theta(\gamma)\big] \ .$$

Since E has no non-zero parallel sections it follows that the 3-form R^E, when considered as a map $E \to \Lambda^2 E$, is injective. Therefore μ is unique. Since the Clifford 4-form Ω is determined by the Clifford structure and this is preserved by $\mathrm{Aut}(\Sigma)$, we have $\mathscr{L}_\gamma \Omega = 0$. A similar calculation to that used in the proof of Theorem 10.4.4, using Lemma B.0.26, shows that $d^{\nabla^E}\big[\Theta(\gamma)\big]$ is in the image of R^E, so μ exists. An explicit formula for μ in this more general setting is not clear. $\square$

In this generalisation the non-vanishing of the scalar curvature has been replaced with the hypothesis that E has no non-zero parallel sections, which we know to be equivalent in the quaternionic-Kähler case by Lemma 10.4.5. By the $\mathrm{Aut}(\Sigma)$-invariance of the defining formula $\nabla^E\big[\mu(\gamma)\big] = \Theta(\gamma)$ it follows from the uniqueness property that μ is $\mathrm{Aut}(\Sigma)$-equivariant.

Consider the action of a group $G < \mathrm{Aut}(\Sigma)$ on Σ. In precise analogy with the quaternionic-Kähler version,

Definition 10.4.13. *The **Clifford moment map** of a G-action is the section μ^G of $\mathfrak{g}^* \otimes E$ given by the restriction of μ in Theorem 10.4.12 to $\mathfrak{g} \subset \mathfrak{aut}(\Sigma)$. We denote its zero set by Z_{μ^G}.*

The G-equivariance of μ^G follows from the $\mathrm{Aut}(\Sigma)$-equivariance of μ. We could now define the Clifford quotient using the moment map μ^G. However, the list of parallel Clifford structures is short and this probably will not lead to any new constructions. However, it does reproduce the quotients we have considered in this chapter.

It would be more interesting to prove an analogous result to Theorem 10.4.12 for even Clifford structures. Then we could apply it to all of the spaces of Theorem 10.4.8, and possibly find a new construction. We could then ask the question: for an appropriate choice of the big space Σ from the list of Theorem 10.4.8, does there exist a G-action on Σ whose reduced phase spaces inherit a parallel non-flat Clifford structure of rank 7?

We do not answer this question in this thesis, and the problem of proving a generalisation of Galicki and Lawson's Theorem 10.4.7 is left open.

Chapter 11

Further Work

We hope that there will be several more uses for Theorem 5.2.4. One such possibility is motivated by Bär's cone construction, in which a Killing spinor field on the base of a cone is used to make a parallel spinor field on the total space of the cone. We may be able to use a similar idea in other situations. Consider the oriented orthonormal frame bundle $\mathcal{M} = SO(\mathcal{B})$ of the Riemannian spin manifold $\mathcal{B}$. The fibres are isometric to SO_n with its bi-invariant metric (of fixed volume), so $\mathcal{M}$ has a natural spin structure compatible with the projection $\pi : \mathcal{M} \to \mathcal{B}$. We have not yet been able to determine if SO_n admits interesting spinor fields. If we assume the base $\mathcal{B}$ does have an interesting spinor field, such as a sort of generalised Killing spinor, then we can construct a spinor field on $\mathcal{M}$ from these vertical and horizontal parts. Theorem 5.2.4 can then be used to calculate its covariant derivative, remembering that O'Neill's tensor A is the curvature of the Levi-Civita connection and $T = 0$ because the fibres are totally geodesic. In this way we may be able to show that a special geometric structure on $\mathcal{B}$ gives us a special geometric structure on $SO(\mathcal{B})$.

Equipping $SO(\mathcal{B})$ with the Sasaki-Mök metric [Mök78] makes $SO(\mathcal{B}) \to \mathcal{B}$ a Riemannian submersion. However, we can consider a conformal rescaling of this metric in order to wake up more of the terms in the formulae of Theorem 5.2.4. We have also attempted such a modification to find more non-trivial examples of Dirac morphisms, but so far to no avail. A more general procedure is to *skew* the Sasaki-Mök metric, i.e. to conformally rescale it only along $\mathscr{H}$ and not $\mathscr{V}$. This has been considered (not in this context) in [KS08b] and [Sek08]. We have shown that this does not produce new Dirac morphisms, and we do not know if special spinor fields can be found. Taking things further still we could skew the metric along $\mathscr{H}$ and along $\mathscr{V}$ by different factors:

$$\tilde{g} = \lambda_{\mathscr{V}} \hat{g} + \lambda_{\mathscr{H}} \check{g} \ .$$

This will affect the covariant derivative in a new way and we have not determined a formula, though one should be simple to derive. The fibres of $SO(\mathcal{B})$ are no longer totally geodesic, and we do not know if this can lead to special spinor fields on the total space or to new Dirac morphisms.

As in Definition 7.2.8 a manifold with G_2-structure has a canonical vector field θ, where the V-part of $d\phi$ is $(1/4)\theta \wedge \phi$. If the G_2-structure has type V

$$d\phi = \frac{1}{4}\theta \wedge \phi$$

and therefore

$$0 = d(\theta \wedge \phi) = d\theta \wedge \phi - \theta \wedge d\phi = d\theta \wedge \phi$$

and $\wedge\phi_0 : \Lambda^2 V \to \Lambda^5 V$ is an isomorphism. This means θ is closed. Similar calculations work for other classes, and Karigiannis [Kar03] lists several of these.

When θ is closed there locally exists a function K such that $\theta = dK$. We could call such a K a *conformal potential* for the structure. It would be interesting to investigate if this can be used as a conformal factor of a submersion, perhaps to skew the Sasaki-Mök metric on the 28-dimensional oriented orthonormal frame bundle $SO(\mathcal{M})$. Then one could try to apply Theorem 5.2.8 to the submersion $SO(\mathcal{M}) \to \mathcal{M}$ in order to construct special spinor fields on the total space, perhaps revealing some geometrical facts about $SO(\mathcal{M})$ in the case $\mathcal{M}$ carries a G_2-structure of type V. There are several problems to overcome, and one would need to find nicely-behaved spinor fields on the fibre SO_7. This last step is not clear, as we have mentioned already.

It would be nice to extend the proof of local existence of associative submanifolds (see [RS09]) to classes of G_2-structures other than 0, $\mathbb{R}$ and $\mathfrak{g}_2$. It may be possible do this when ϕ is coclosed, i.e. to types in $\mathbb{R} \oplus S^2_0 V \oplus \mathfrak{g}_2$. Global existence of associatives is a different question and possibly has a different answer. Up to this point the author has seen no proof of local or global existence of Cayley submanifolds when the $Spin_7^+$-structure has type F.

The formula for the mean curvature of associative submanifolds suggests they are not in general minimal, if they do exist for other classes. We do not expect the mean curvature flow to preserve associativity, but perhaps the stationary points are interesting. The same questions can be asked for Cayley submanifolds in the type F and generic cases of $Spin_7^+$-structure.

A quotient construction yielding Riemannian manifolds with $Spin_7$ holonomy still eludes us. To find one, it would be necessary to extend the notions of Clifford 4-form and Clifford moment map to even Clifford structures. This allows greater choice for the big space Σ and thus more possible new quotients. This may not be entirely straightforward because an even Clifford structure E cannot be considered as a subbundle of $\Lambda^2 T\Sigma$, and this makes the situation less clear. However, with suitable modifications, the author believes a quotient can be defined. It would be very interesting to find out which geometries arise in this way.

Appendix A

Octonionic Multiplication

The octonions $\mathbb{O}$ are the algebra given by the Cayley-Dickson construction on the quaternions $\mathbb{H}$. That is, $\mathbb{O} = \mathbb{H} + \mathbb{H}$ with

$$(a, b)(c, d) \stackrel{\text{def}}{=} (ac - \bar{d}b, da + b\bar{c})$$

where $^{-}$ is quaternionic conjugation. In this language octonionic conjugation is given by $\overline{(a, b)} = (\bar{a}, -b)$. Since we know how to multiply quaternions we can write a multiplication table for octonions. We'll use the basis $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ for the imaginary quaternions, and gothic numbers for the imaginary octonions:

$$1 = (1, 0)\ ,\ \mathfrak{1} = (\mathfrak{i}, 0)\ ,\ \mathfrak{2} = (\mathfrak{j}, 0)\ ,\ \mathfrak{3} = (\mathfrak{k}, 0)$$
$$\mathfrak{4} = (0, 1)\ ,\ \mathfrak{5} = (0, \mathfrak{i})\ ,\ \mathfrak{6} = (0, \mathfrak{j})\ ,\ \mathfrak{7} = (0, \mathfrak{k})\ .$$

We have to be careful to remember that $\mathfrak{1}$ does not denote the multiplicative identity. Now

	$\mathfrak{1}$	$\mathfrak{2}$	$\mathfrak{3}$	$\mathfrak{4}$	$\mathfrak{5}$	$\mathfrak{6}$	$\mathfrak{7}$
$\mathfrak{1}$	-1	$\mathfrak{3}$	$-\mathfrak{2}$	$\mathfrak{5}$	$-\mathfrak{4}$	$-\mathfrak{7}$	$\mathfrak{6}$
$\mathfrak{2}$	$-\mathfrak{3}$	-1	$\mathfrak{1}$	$\mathfrak{6}$	$\mathfrak{7}$	$-\mathfrak{4}$	$-\mathfrak{5}$
$\mathfrak{3}$	$\mathfrak{2}$	$-\mathfrak{1}$	-1	$\mathfrak{7}$	$-\mathfrak{6}$	$\mathfrak{5}$	$-\mathfrak{4}$
$\mathfrak{4}$	$-\mathfrak{5}$	$-\mathfrak{6}$	$-\mathfrak{7}$	-1	$\mathfrak{1}$	$\mathfrak{2}$	$\mathfrak{3}$
$\mathfrak{5}$	$\mathfrak{4}$	$-\mathfrak{7}$	$\mathfrak{6}$	$-\mathfrak{1}$	-1	$-\mathfrak{3}$	$\mathfrak{2}$
$\mathfrak{6}$	$\mathfrak{7}$	$\mathfrak{4}$	$-\mathfrak{5}$	$-\mathfrak{2}$	$\mathfrak{3}$	-1	$-\mathfrak{1}$
$\mathfrak{7}$	$-\mathfrak{6}$	$\mathfrak{5}$	$\mathfrak{4}$	$-\mathfrak{3}$	$-\mathfrak{2}$	$\mathfrak{1}$	-1

Octonionic multiplication is non-associative, so to cut down on brackets we shall adopt the notation

$$\underleftarrow{xyz} \stackrel{\text{def}}{=} x(yz)$$

and in general

$$\underleftarrow{w \cdots xyz} \stackrel{\text{def}}{=} w(\cdots(x(yz)))\ .$$

A quick calculation using the table above shows that when x, y, z are mutually orthogonal elements of $\mathrm{Im}\mathbb{O}$, $\underleftarrow{xyz}$ is cyclic. Note that this doesn't hold if we allow non-imaginary elements, or if x, y, z are not mutually orthogonal.

We now look at some facts which the author found particularly useful for calculations (although we may use them implicitly without reference).

Lemma A.0.14. *Let x, y, z, w be mutually orthogonal elements of* $\mathrm{Im}\mathbb{O}$. *Then* $\underleftarrow{xyzw}$ *is anticyclic.*

Proof. Since $y(zw)$ is cyclic,

$$x\big[y(zw)\big] = x\big[w(yz)\big] \ .$$

Note that $w||yz \iff z||wy \iff y||zw$. There are two cases:

1. $w||yz$. Then since $\mathbb{O}$ is an alternative algebra (i.e. the subalgebra generated by any pair of elements is associative), we have that

$$x\big[w(yz)\big] = (xw)(yz) = -(wx)(yz) = -w\big[x(yz)\big] \ .$$

2. $w \perp yz$. Then

$$x\big[w(yz)\big] = -x\big[(yz)w\big] \ .$$

   This case splits into a further two cases; if $x||yz$ then by the alternative property

$$-x\big[(yz)w\big] = -\big[x(yz)\big]w = -w\big[x(yz)\big] \ ,$$

   and if $x \perp yz$ then

$$-x\big[(yz)w\big] = -w\big[x(yz)\big]$$

   by cyclicity.

□

Lemma A.0.15. *Let x, y, z, w, t, r, s be mutually orthogonal elements of* $\mathrm{Im}\mathbb{O}$. *Then*

1. $\underleftarrow{xyzwt}$ *is cyclic*
2. $\underleftarrow{xyzwtr}$ *is anticyclic.*
3. $\underleftarrow{xyzwtrs}$ *is cyclic.*

Proof. Similar to above. □

Appendix B

Linear Algebra

Lemma B.0.16. *In dimension n, for an $(n-p)$-form α and a p-form x*

$$\alpha(*x) = (-1)^{p(n-1)} \frac{p!}{(n-p)!} * \alpha(x) \ .$$

Proof. Choose an orthonormal frame $\mathfrak{1}, \ldots, \mathfrak{n}$ in which we can write $\alpha = \sum_{\mathfrak{i}_1,\ldots,\mathfrak{i}_p} \alpha_{\mathfrak{i}_1\cdots\mathfrak{i}_p} * (\mathfrak{i}_1 \wedge \ldots \wedge \mathfrak{i}_p)$. Then

$$*\alpha = \sum_{\mathfrak{i}_1,\ldots,\mathfrak{i}_p} \alpha_{\mathfrak{i}_1\cdots\mathfrak{i}_p} * *(\mathfrak{i}_1 \wedge \ldots \wedge \mathfrak{i}_p) = \sum_{\mathfrak{i}_1,\ldots,\mathfrak{i}_p} \alpha_{\mathfrak{i}_1\cdots\mathfrak{i}_p} (-1)^{p(n-1)} (\mathfrak{i}_1 \wedge \ldots \wedge \mathfrak{i}_p) \ .$$

Without loss of generality we can put $x = \mathfrak{1} \wedge \ldots \wedge \mathfrak{p}$, and then

$$\alpha(*x) = p! \alpha_{\mathfrak{1}\cdots\mathfrak{p}} * (\mathfrak{1} \wedge \ldots \wedge \mathfrak{p})(*(\mathfrak{1} \wedge \ldots \wedge \mathfrak{p})) = p! \alpha_{\mathfrak{1}\cdots\mathfrak{p}} \frac{1}{(n-p)!}$$

and

$$*\alpha(x) = p! \alpha_{\mathfrak{1}\cdots\mathfrak{p}} \mathfrak{1} \wedge \ldots \wedge \mathfrak{p}(\mathfrak{1} \wedge \ldots \wedge \mathfrak{p}) = (-1)^{p(n-1)} p! \alpha_{\mathfrak{1}\cdots\mathfrak{p}} \frac{1}{p!} \ .$$

The result is now clear. □

Lemma B.0.17. *In dimension n, for a p-form α and vector v*

$$\iota_v \alpha = (-1)^{n(p+1)} \frac{1}{p} * (v \wedge *\alpha) \ .$$

Proof. Similar to above. □

Lemma B.0.18. *In dimension n, for a p-form α and 2-form β*

$$\iota_\beta \alpha = (-1)^{p(n+1)} \frac{1}{p(p-1)} * (\beta \wedge *\alpha)$$

and for β a 3-form

$$\iota_\beta \alpha = (-1)^{n(p+1)} \frac{1}{p(p-1)(p-2)} * (\beta \wedge *\alpha)$$

and for β a 4-form

$$\iota_\beta \alpha = (-1)^{p(n+1)} \frac{1}{p(p-1)(p-2)(p-3)} * (\beta \wedge *\alpha) \ .$$

Proof. These follow from repeated application of Lemma B.0.17. ∎

Lemma B.0.19. *For ϕ_0 the canonical 3-form on $V = \mathrm{Im}\mathbb{O}$,*

$$\phi_0(x, \phi_0(x, y, \cdot), \cdot) = -\frac{1}{36} g(x, x) y \ .$$

Proof. Firstly

$$\phi_0(x, \phi_0(x, y, \cdot), \cdot) = -\frac{1}{6}\mathrm{Re}\big[\underrightarrow{x\phi_0(x, y, \cdot)\ \cdot}\,\big] = \frac{1}{36}\mathrm{Re}\big[\underrightarrow{x[\underrightarrow{\mathrm{Re}(xy)\ \cdot}\,]\ \cdot}\ \big] \ .$$

Using the basis $1, \dots, 7$ we can check that $\mathrm{Re}\underrightarrow{xy\ \cdot}$ is equal to minus the dual of the vector (xy), so that

$$x(\mathrm{Re}\underrightarrow{xy\ \cdot}\,) = -x(xy) = -(xx)y = g(x, x)y$$

using the alternative property. Then

$$\phi_0(x, \phi_0(x, y, \cdot), \cdot) = \frac{1}{36}\mathrm{Re}\big[g(x, x)y\ \cdot\big] = \frac{1}{36} g(x, x)\mathrm{Re}(y\ \cdot\ ) = -\frac{1}{36} g(x, x)y \ .$$

∎

Note that this lemma immediately implies a lemma of [FG82]:

$$\phi_0(x, \phi_0(x, \phi_0(x, y, \cdot), \cdot), \cdot) = -\frac{1}{36} g(x, x)\phi_0(x, y, \cdot) \ .$$

Lemma B.0.20. *For any vectors x, y, z in $V = \mathrm{Im}\mathbb{O}$*

$$g(x, y)z = -18\big[\phi_0(x \wedge \phi_0(y \wedge z)) + \phi_0(y \wedge \phi_0(x \wedge z))\big]$$

and in particular, when z is any unit vector, the inner product on $V = \mathrm{Im}\mathbb{O}$ may be written as

$$g(x, y) = 18\big[\phi_0(x \wedge z \wedge \phi_0(y \wedge z)) + \phi_0(y \wedge z \wedge \phi_0(x \wedge z))\big] \ .$$

Proof. These follow by polarisation of B.0.19. ∎

Lemma B.0.21. *For any vectors x, y and unit vectors w, z, the inner product on $W = \mathbb{O}$ may be written as*

$$g(x, y) = 288\big[\Phi_0(w \wedge x \wedge z \wedge \Phi_0(w \wedge y \wedge z)) + \Phi_0(w \wedge y \wedge z \wedge \Phi_0(w \wedge x \wedge z))\big] \ .$$

Proof. This is similar to the case for ϕ_0. ∎

Lemma B.0.22. *The following formulae hold for $x, y, z, u, v, w \in V = \mathrm{Im}\mathbb{O}$, where $[\cdot, \cdot]$ is the commutator, $\{\cdot, \cdot\}$ is the anticommutator and $\mathfrak{R}$ is the Clifford representation introduced in Definition 7.3.1.*

1. $[\mathfrak{R}(x), \mathfrak{R}(y)] = 2\mathfrak{R}(x \wedge y)$
2. $\{\mathfrak{R}(x), \mathfrak{R}(y)\} = -2g(x, y)$
3. $[\mathfrak{R}(x \wedge y), \mathfrak{R}(z)] = 2g(x, z)\mathfrak{R}(y) - 2g(y, z)\mathfrak{R}(x)$

4. $\{\mathfrak{R}(x \wedge y), \mathfrak{R}(z)\} = 2\mathfrak{R}(x \wedge y \wedge z)$

5. $[\mathfrak{R}(x \wedge y \wedge z), \mathfrak{R}(w)] = 2\mathfrak{R}(x \wedge y \wedge z \wedge w)$

6. $\{\mathfrak{R}(x \wedge y \wedge z), \mathfrak{R}(w)\} = -2\mathfrak{R}\big(g(z,w)x \wedge y + g(y,w)z \wedge x + g(x,w)y \wedge z\big)$

7.
$$\begin{aligned}[\mathfrak{R}(w \wedge x), \mathfrak{R}(y \wedge z)] = -2\mathfrak{R}\big(&-g(w,z)y \wedge x - g(w,y)x \wedge z \\ &+ g(x,z)y \wedge w + g(x,y)w \wedge z + 2g(x,y)g(w,z) - 2g(w,y)g(x,z)\big) \\ = -4\mathfrak{R}\big(&(w \wedge x)_*(y \wedge z) + g(w \wedge x, y \wedge z)\big)\end{aligned}$$

where $w \wedge x$ acts on $y \wedge z$ by its Lie algebra action (of $\mathfrak{so}_7$ on $\Lambda^2 V$, i.e. the action of $\Lambda^2 V$ on V extended to an action on $\Lambda^2 V$).

8. $\{\mathfrak{R}(w \wedge x), \mathfrak{R}(y \wedge z)\} = 2\mathfrak{R}(w \wedge x \wedge y \wedge z)$

9.
$$\begin{aligned}[\mathfrak{R}(x \wedge y), \mathfrak{R}(u \wedge v \wedge w)] = 2\mathfrak{R}\big(&-g(y,u)x \wedge v \wedge w + g(x,u)y \wedge v \wedge w \\ &- g(y,v)u \wedge x \wedge w + g(x,v)u \wedge y \wedge w \\ &- g(y,w)u \wedge v \wedge x + g(x,w)u \wedge v \wedge y\big) \\ = -4\mathfrak{R}\big(&(x \wedge y)_*(u \wedge v \wedge w)\big)\end{aligned}$$

where $x \wedge y$ acts on $u \wedge v \wedge w$ by its Lie algebra action (of $\mathfrak{so}_7$ on $\Lambda^3 V$, i.e. the action of $\Lambda^2 V$ on V extended to an action on $\Lambda^3 V$).

10.
$$\begin{aligned}\{\mathfrak{R}(x \wedge y), \mathfrak{R}(u \wedge v \wedge w)\} = 2\mathfrak{R}\big(&x \wedge y \wedge u \wedge v \wedge w \\ &+ g(y,u)g(x,v)w - g(y,u)g(x,w)v - g(x,u)g(y,v)w \\ &+ g(x,u)g(y,w)v + g(y,v)g(x,w)u - g(x,v)g(y,w)u\big)\end{aligned}$$

11.
$$\begin{aligned}[\mathfrak{R}(x \wedge y \wedge z), \mathfrak{R}(u \wedge v \wedge w)] = 2\mathfrak{R}\big(&x \wedge y \wedge z \wedge u \wedge v \wedge w \\ &+ g(z,u)g(y,v)x \wedge w - g(z,u)g(x,v)y \wedge w - g(z,u)g(y,w)x \wedge v \\ &+ g(z,u)g(x,w)y \wedge v - g(y,u)g(z,v)x \wedge w + g(y,u)g(x,v)z \wedge w \\ &+ g(y,u)g(z,w)x \wedge v - g(y,u)g(x,w)z \wedge v + g(x,u)g(z,v)y \wedge w \\ &- g(x,u)g(y,v)z \wedge w - g(x,u)g(z,w)y \wedge v + g(x,u)g(y,w)z \wedge v \\ &+ g(z,v)g(y,w)x \wedge u - g(z,v)g(x,w)y \wedge u - g(y,v)g(z,w)x \wedge u \\ &+ g(y,v)g(x,w)z \wedge u + g(x,v)g(z,w)y \wedge u - g(x,v)g(y,w)z \wedge u\big)\end{aligned}$$

12.

$$
\begin{aligned}
\{\mathfrak{R}(x\wedge y\wedge z),\mathfrak{R}(u\wedge v\wedge w)\} &= 2\mathfrak{R}\big(-g(z,u)x\wedge y\wedge v\wedge w\\
&+g(y,u)x\wedge z\wedge v\wedge w-g(x,u)y\wedge z\wedge v\wedge w+g(z,v)x\wedge y\wedge u\wedge w\\
&-g(y,v)x\wedge z\wedge u\wedge w+g(x,v)y\wedge z\wedge u\wedge w-g(z,w)x\wedge y\wedge u\wedge v\\
&+g(y,w)x\wedge z\wedge u\wedge v-g(x,w)y\wedge z\wedge u\wedge v\\
&-g(z,u)g(y,v)g(x,w)+g(z,u)g(x,v)g(y,w)+g(y,u)g(z,v)g(x,w)\\
&-g(y,u)g(x,v)g(z,w)-g(x,u)g(z,v)g(y,w)+g(x,u)g(y,v)g(z,w)\big)\\
\\
&=-6\mathfrak{R}\big((x\wedge y\wedge z)_*(u\wedge v\wedge w)-g(x\wedge y\wedge z,u\wedge v\wedge w)\big)
\end{aligned}
$$

where $(x\wedge y\wedge z)$ acts on $(u\wedge v\wedge w)$ by the action of Λ^3V on V via g extended to an action on Λ^3V.

13. *Now where the vectors lie in $W=\mathbb{O}$,*

$$
\begin{aligned}
[\mathfrak{R}(x\wedge y),\mathfrak{R}(t\wedge u\wedge v\wedge w)] &= 2\mathfrak{R}\big(-g(y,t)x\wedge u\wedge v\wedge w\\
&+g(y,u)x\wedge t\wedge v\wedge w-g(y,v)x\wedge t\wedge u\wedge w+g(y,w)x\wedge t\wedge u\wedge v\\
&+g(x,t)y\wedge u\wedge v\wedge w-g(x,u)y\wedge t\wedge v\wedge w+g(x,v)y\wedge t\wedge u\wedge w\\
&-g(x,w)y\wedge t\wedge u\wedge v\big)\\
\\
&=4\mathfrak{R}\big((x\wedge y)_*(t\wedge u\wedge v\wedge w)\big)
\end{aligned}
$$

where $x\wedge y$ acts on $t\wedge u\wedge v\wedge w$ by its Lie algebra action (of $\mathfrak{so}_8$ on Λ^4W, i.e. the action of Λ^2W on W extended to an action on Λ^4W). Actually we have used a hidden abuse of notation: the above formula is valid also when one of the vectors is o, which is our new name for $1\in\mathbb{O}$ when in eight dimensions. Of course $\mathfrak{R}(\mathsf{o})$ makes no sense but in the above is the simplest way of including o.

14.

$$
\begin{aligned}
\{\mathfrak{R}(w\wedge x\wedge y\wedge z),&\mathfrak{R}(s\wedge t\wedge u\wedge v)\}\\
&=2\mathfrak{R}\big(w\wedge x\wedge y\wedge z\wedge s\wedge t\wedge u\wedge v\big)\\
&-36\mathfrak{R}\big((w\wedge x\wedge y\wedge z)\star_{1,2}(s\wedge t\wedge u\wedge v)\big)\\
&+4!g(w\wedge x\wedge y\wedge z,s\wedge t\wedge u\wedge v)
\end{aligned}
$$

where $\star_{1,2}$ is contraction over the first two indices of the two 4-forms. The same comments about o made in the point above also apply here.

15. $\sum_{\mathsf{i}=1}^{7}\mathfrak{R}(\mathsf{i})\mathfrak{R}(x\wedge y\wedge z)\mathfrak{R}(\mathsf{i})=\mathfrak{R}(x\wedge y\wedge z)$

Proof. Lengthy calculation. □

Lemma B.0.23. *The Hodge star $*$ acts on ΛV, which for $V=\mathrm{Im}\mathbb{O}$ is naturally isomorphic as a vector space to $C\ell_7$. With respect to this isomorphism and where $(\mathfrak{R},\Delta)$ is the real Clifford* *representation, for* $\alpha \in \Lambda^k V$,

$$\Re(*\alpha) = \begin{cases} \Re(\alpha) & \text{for } k = 0,3,4,7 \\ -\Re(\alpha) & \text{for } k = 1,2,5,6 \end{cases}$$

Proof. Direct calculation. □

Proposition B.0.24. *Let* $*_7$ *and* $*_8$ *denote the Hodge star operators in seven and eight dimensions, acting respectively on* $V = \mathrm{Im}\mathbb{O}$ *and* $W = \mathbb{O}$*. Writing*

$$W = \mathbb{R}\mathtt{0} \oplus V$$

allows us an embedding $\Lambda V \subset \Lambda W$*, where* ΛV *consists of those forms which do not contain any* $\mathtt{0}$*'s. Let* α *and* β *be forms not containing any* $\mathtt{0}$*'s. Then*

$$*_8\alpha = (-1)^{|\alpha|}\mathtt{0} \wedge *_7\alpha$$

and

$$*_8(\mathtt{0} \wedge \beta) = *_7\beta \ .$$

Proof. Calculation, similar to Proposition B.0.25 below. □

Proposition B.0.25. *Suppose we have a warped product* $\mathbb{R} \times_{f^2} \mathcal{M}$ *and we want to know how the Hodge star operator* $*_8$ *of this space is related to that of* $\mathcal{M}$*. We can do this at a point, and the result is a generalisation of Proposition B.0.24. This time the tangent space of the product looks like* $W = \mathbb{O}$ *but now with inner product*

$$\langle\ \cdot\ ,\ \cdot\ \rangle_W = \mathtt{0} \otimes \mathtt{0} + f^2 \langle\ \cdot,\ \cdot\ \rangle_V \ .$$

We claim that for α *and* β *as in Proposition B.0.24,*

$$*_8\alpha = (-1)^{|\alpha|} f^{2|\alpha|-7}\mathtt{0} \wedge *_7\alpha$$

and

$$*_8(\mathtt{0} \wedge \beta) = f^{2|\beta|-7} *_7 \beta \ .$$

Proof. An othornormal frame of W with this inner product is

$$\mathtt{0}, f^{-1}\mathtt{1}, \ldots, f^{-1}\mathtt{7}$$

where $\mathtt{1}, \ldots, \mathtt{7}$ is an orthonormal frame of V. Now it is just a matter of checking the claim on simple pieces:

$$*_8 f^{-1}\mathtt{1} = -f^{-6}\mathtt{0234567}$$

(where we omit the $\wedge$'s) so

$$*_8\mathtt{1} = -f^{-5}\mathtt{0234567} = -f^{-5}\mathtt{0} \wedge *_7\mathtt{1} \ .$$

Also

$$*_8 f^{-2}\mathtt{12} = f^{-5}\mathtt{034567}$$

so

$$*_8 12 = f^{-3} 034567 = f^{-3} 0 \wedge *_7 12 \ .$$

The others are similar. For the second part

$$*_8 0 = f^{-7} 1234567 = f^{-7} *_7 1$$

where this 1 is the identity in $\mathbb{O}$. Also

$$*_8 f^{-1} 01 = f^{-6} 234567$$

so

$$*_8 01 = f^{-5} *_7 1$$

and so on. □

We know that $*_7 *_7 = 1$. Since any k-form in ΛW can be written as $\alpha + 0 \wedge \beta$ where α and β don't contain any 0's, we can show

$$*_8 *_8 \left[\alpha + 0 \wedge \beta\right] = (-1)^k \left[\alpha + 0 \wedge \beta\right]$$

(all the f's conveniently cancel). This confirms $*_8 *_8 = (-1)^k$ on k-forms.
In the above proposition we used implicitly the isomorphisms

$$\Lambda^{k-1} V \oplus \Lambda^k V \xrightarrow{\sim} \Lambda^k W \ : \ (\alpha, \beta) \to 0 \wedge f^{-(k-1)} \alpha + f^{-k} \beta \ .$$

Lemma B.0.26. *Let V be a real vector space of dimension $2n > 4$, and let $J_1, \ldots, J_p$ be a set of anticommuting complex structures on V. Suppose $\alpha_1, \ldots, \alpha_p$ are elements of $\Lambda^2 V$ such that*

$$\sum_{i=1}^{p} \alpha_i \wedge J_i = 0 \ .$$

Then we may write

$$\alpha_i = \sum_{j=1}^{p} \alpha_{ij} J_j \ , \ \alpha_{ji} = -\alpha_{ij} \ .$$

Proof. It is always possible to choose a basis of V so that the complex structure J_1 is given by

$$J_1 = 2e_1 \wedge e_2 + \cdots + 2e_{2n-1} \wedge e_{2n} \ .$$

Suppose an element $\beta = \sum_{i<j} \beta_{ij} e_i \wedge e_j \in \Lambda^2 V$ satisfies $\beta \wedge J_1 = 0$. If $2n = 4$ then

$$0 = \beta \wedge J_1 = 2 \sum_{i<j} \beta_{ij} e_i \wedge e_j \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)$$

is equivalent to the single condition $\beta_{12} + \beta_{34} = 0$. The subspace of the 6-dimensional $\Lambda^2 V$ of forms β such that $\beta \wedge J_1 = 0$ is therefore 5-dimensional. If $2n > 4$ then in the expression for J_1 we now have the crucial term $2e_5 \wedge e_6$. From

$$0 = \beta \wedge J_1 = 2 \sum_{i<j} \beta_{ij} e_i \wedge e_j \wedge (e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n})$$

we see that $\beta_{ij} = 0$ whenever the pair (i, j) is not one of $(1, 2), (3, 4), \ldots, (2n-1, 2n)$. We also see that

$$\beta_{34} = -\beta_{12} \ , \ \beta_{56} = -\beta_{12} \ , \ \beta_{56} = -\beta_{34}$$

from which it follows that $\beta_{12} = 0$, and similarly $\beta_{34} = \beta_{56} = \cdots = \beta_{2n-1\ 2n} = 0$. Thus $\beta = 0$ and the subspace of $\Lambda^2 V$ of forms β such that $\beta \wedge J_1 = 0$ is trivial. Notice the contrast with the case $2n = 4$.

Using this fact, we see that the wedge products between all the complex structures $J_1, \ldots, J_p$ are non-vanishing when $2n > 4$. Moreover, we can extend this set of complex structures to a basis $J_1, \ldots, J_p, \beta_{p+1}, \ldots, \beta_{n(2n-1)}$ of $\Lambda^2 V$ such that $\beta_i \wedge J_j \neq 0$ for all i, j. The β_i's need not be complex structures (they usually cannot be) and we make no requirement on $\beta_i \wedge \beta_j$. Using this basis we can write

$$\alpha_i = \sum_{j=1}^{p} \alpha_{ij} J_j + \sum_{j=p+1}^{n(2n-1)} \alpha_{ij} \beta_j$$

and then

$$0 = \sum_{i=1}^{p} \alpha_i \wedge J_i = \sum_{i,j=1}^{p} \alpha_{ij} J_j \wedge J_i + \sum_{i=1}^{p} \sum_{j=p+1}^{n(2n-1)} \alpha_{ij} \beta_j \wedge J_i \ .$$

It follows that $\alpha_{ij} = 0$ for $p + 1 \leq j \leq n(2n-1)$ and

$$\alpha_i = \sum_{j=1}^{p} \alpha_{ij} J_j \ , \ \alpha_{ji} = -\alpha_{ij} \ .$$

□

Appendix C

Cross Products, Cohomology and Harmonicity

Cross products were defined by Eckmann as follows.

Definition C.0.27. *([Eck43]) An r**-fold cross product** on a vector space V^n with bilinear form g is a continuous map*

$$P : V^{\otimes r} \to V$$

satisfying

1. *P is skew;*

$$g\big(P(v_1, \dots, v_r), v_i\big) = 0 \ , \ \ 1 \leq i \leq r \ ,$$

2. *P respects g;*

$$g\big(P(v_1, \dots, v_r), P(v_1, \dots, v_r)\big) = (r+1)!^2 \det g(v_i, v_j) \ .$$

   We shall also require the additional axiom

3. *P is linear*

   although Eckmann did not include this.

We may also use the more familiar notation when $r = 2$

$$v_1 \times v_2 \overset{\text{def}}{=} P(v_1, v_2)$$

and we can even do the same for higher values of r, for example if $r = 3$

$$v_1 \times v_2 \times v_3 \overset{\text{def}}{=} P(v_1, v_2, v_3) \ .$$

If P is linear then *1* allows us to write $P : \Lambda^r V \to V$. Then *2* just means that P preserves norms of simple elements:

$$||P(v_1 \wedge \ldots \wedge v_r)|| = ||v_1 \wedge \ldots \wedge v_r|| \quad \forall v_1, \ldots, v_r \in V \ .$$

It does not follows that P preserves norms of all elements.

There are several familiar examples.

1. $r = 0$: The conditions of Definition C.0.27 are vacuous and P is just a choice of element of V.

2. $r = 1$: Then
$$g(P(v), v) = 0 \quad \forall v$$
and
$$g(P(v), P(v)) = g(v, v) \quad \forall v$$
so P is a complex structure on V compatible with g.

3. $r = n - 1$: Then P is an n-form on V. It is proportional to the volume form of g with a chosen orientation (so P merely gives us an orientation).

The following theorem was proven by Eckmann and Whitehead using methods from algebraic topology. In the case P is linear, the same theorem was proven by Brown and Gray.

Theorem C.0.28. *([Eck43, Whi63], [BG67]) An r-fold cross product on a real vector space V^n exists if and only if*

1. *n even, $r = 1$,*

2. *$n = 7$, $r = 2$,*

3. *$n = 8$, $r = 3$,*

4. *n arbitrary, $r = n - 1$.*

Eckmann and Whitehead reformulated cross products as section of fibrations $S_{n,m} \to S_{n,r}$, where $S_{n,m}$ is the Stiefel manifold of m-frames in n-dimensions.
Brown and Gray's proof of the theorem is algebraic, using

Definition C.0.29. *A* ***composition algebra*** (W, N) *is a finite-dimensional algebra W (possibly non-associative) with a quadratic form N such that*

1. *W is unital,*

2. *$\langle x, y\rangle \overset{\text{def}}{=} \frac{1}{2}\big[N(x + y) - N(x) - N(y)\big]$ is non-degenerate,*

3. *N is multiplicative: $N(xy) = N(x)N(y) \quad \forall x, y \in W$.*

In the case $r = 2$, the next step in the proof is

Lemma C.0.30. *([BG67])*

1. *Let W be a composition algebra and let $V \subset W$ be the orthogonal complement of the multiplicative identity e. Then*
$$P : V \times V \to V : P(x, y) \overset{\text{def}}{=} xy + \langle x, y\rangle e$$
is a 2-fold cross product on V with $g = \langle\ \cdot\ ,\ \cdot\ \rangle|_V$.

2. *Let P be a 2-fold cross product on V and put $W = \mathbb{R}e \oplus V$. Extend the bilinear form g on V to W by*
$$g(e,e) = 1 \ , \quad g(e,V) = 0 \ .$$
Define a product
$$xy \stackrel{\text{def}}{=} P(x,y) - g(x,y)e \ .$$
Then (W,g) is a composition algebra.

This lemma is proved by a calculation. Finally, we need the structure theorem for composition algebras, originally proved by Hurwitz, that says that a composition algebra has dimension 1, 2, 4 or 8 and is built from the base field by the Cayley-Dickson construction. When the norm N is positive-definite the only real ones are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Part *2* of Theorem C.0.28 follows, and the other cases are similar or easier.

Remark C.0.31. *One may show, using Lemma C.0.30 and the structure theorem for composition algebras, that the automorphisms of a 2-fold cross product on a 7-dimensional real vector space are the same as the real algebra automorphisms of $\mathbb{O}$, i.e. G_2.*

The application of this concept to manifolds was first considered by Gray [Gra69][1].

Definition C.0.32. *([Gra69]) A* ***cross product*** *on a Riemannian manifold $(\mathcal{M}, g)$ is a smooth cross product on $T\mathcal{M}$.*

By Remark C.0.31 it is now clear that a 1-fold cross product on $\mathcal{M}$ is precisely an almost-Hermitian structure; a 2-fold cross product on $\mathcal{M}$ is precisely a G_2-structure; a 3-fold cross product on $\mathcal{M}$ is precisely a $Spin_7^+$-structure; an $(n-1)$-fold cross product on $\mathcal{M}$ is precisely an orientation. Such a cross product P is a tensor field, and linearity allows us to consider P as an $(r+1)$-form on $\mathcal{M}$. These are

1. n even, $r = 1$: the almost-symplectic form ω,
2. $n = 7$, $r = 2$: the canonical form φ,
3. $n = 8$, $r = 3$: the canonical form Φ,
4. n arbitrary, $r = n-1$: the volume form.

The topological obstructions to the existence of cross products in the second and third cases are therefore given by Theorems 7.2.2 and 8.2.2.
As Gray [Gra69] comments, cross products are interesting because they generalise complex structures (we use this perspective in Chapter 9 frequently) and also provide a different approach to manifolds with G_2 or $Spin_7$ holonomy. Obviously,

Theorem C.0.33. *If a 7-dimensional Riemannian manifold has a* parallel *2-fold cross product then it has restricted holonomy contained in G_2. Similarly for $Spin_7$ in 8 dimensions.*

There are several cone constructions which relate cross products of different degrees. Amongst these,

Theorem C.0.34. *([Gra69]) Suppose the sphere S^n has an r-fold cross product. Then the vector space $\mathbb{R}^{n+1}$ has an $(r+1)$-fold cross product (but it may not be linear).*

[1] The existence of [Gra70] is noted here, but it does not affect us.

Proof. Embed S^n into $\mathbb{R}^{n+1}$ in the standard way. For $v_1, \ldots, v_r, v_{r+1} \in \mathbb{R}^{n+1}$, write $v_{r+1} = b+c$ where $b \perp span\{v_1, \ldots, v_r\}$. If P_x is the cross product at $x \in S^n$, one may check that

$$Q(v_1, \ldots, v_r, v_{r+1}) \overset{\text{def}}{=} ||b|| P_{b/||b||}(v_1, \ldots, v_r)$$

defines a cross product on $\mathbb{R}^{n+1}$. □

This has the excellent corollary:

Corollary C.0.35. *([Gra69]) The only round spheres with compatible almost-complex structures are S^2 and S^6.*

Proof. This follows from Lemma C.0.30 and Theorem C.0.34. □

Given any almost-complex structure on the manifold S^n one may choose a metric which makes it orthogonal and skew, and therefore one may construct an almost-Hermitian structure on S^n. We can use this fact to show that the 'round' hypothesis is not necessary in Corollary C.0.35; S^2 and S^6 are the only spheres admitting almost-complex structures.

There are many more interesting consequences of the existence of 2-fold cross products on 7-dimensional Riemannian spin manifolds. Given a cross product we may define the *curl operator* in the same way as usual; for a vector field X

$$curl : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) : X \to curl X \overset{\text{def}}{=} P(\nabla X) = P(dX) \ .$$

Properties of this operator are discussed by Peng and Yang [PY99], and in the excellent review article of Karigiannis [Kar10]. For simplicity, denote by π_V and $\pi_{\mathfrak{g}_2}$ the projections onto the two irreducible subbundles of $\Lambda^2 T\mathcal{M}$ and π_ϕ that onto the ϕ-part of a 3-form. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C^\infty(\mathcal{M}) \arrow[r, "d"] \arrow[d, "="] & \Omega^1(\mathcal{M}) \arrow[r, "\pi_V \circ d"] \arrow[d, "g(\cdot)"] & \pi_V \Omega^2(\mathcal{M}) \arrow[r, "\pi_\phi \circ d"] \arrow[d, "6\phi(\cdot)"] & C^\infty(\mathcal{M})\phi \arrow[r] \arrow[d, "(6/7)g(\phi{,}\cdot)"] & 0 \arrow[d] \\
0 \arrow[r] & C^\infty(\mathcal{M}) \arrow[r, "grad"] & \mathfrak{X}(\mathcal{M}) \arrow[r, "curl"] & \mathfrak{X}(\mathcal{M}) \arrow[r, "div"] & C^\infty(\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

where $\Omega^k(\mathcal{M})$ is the space of k-forms. The lower sequence is considered (somewhat implicitly) by Karigiannis [Kar10]. We have

$$curl \circ grad(f) = 6\phi(ddf) = 0$$

for any $f \in C^\infty(\mathcal{M})$, and

$$\begin{aligned}
div \circ curl(X) &= -\delta\big[6\phi(dX)\big] \\
&= -\delta\big[* (dX \wedge *\phi)\big] \\
&= *d * \big[* (dX \wedge *\phi)\big] \\
&= *d(dX \wedge *\phi) \\
&= *(dX \wedge d * \phi) \\
&= -3 * (dX \wedge *\delta\phi) \\
&= -6\delta\phi(dX)
\end{aligned}$$

for any $X \in \mathfrak{X}(\mathcal{M})$. It follows that the lower sequence is a complex if and only if the G_2-structure is cocalibrated; $\delta\phi = 0$.

The upper sequence was first written down by Salamon [Sal89], but he only proved that it is a complex when $\delta\phi = 0$. In fact, as pointed out by Fernández and Ugarte [FU98, FU00], this sequence is a complex for more classes. Noting that

$$d\big[\pi_{\mathfrak{g}_2}(dX)\big] \wedge *\phi = -\pi_{\mathfrak{g}_2}(dX) \wedge d * \phi = 3\pi_{\mathfrak{g}_2}(dX) \wedge *\pi_{\mathfrak{g}_2}(\delta\phi)$$

we have

$$\begin{aligned}
(\pi_\phi \circ d) \circ (\pi_V \circ d)(X) &= (\pi_\phi \circ d)(dX - \pi_{\mathfrak{g}_2}(dX)) \\
&= -\pi_\phi \circ d(\pi_{\mathfrak{g}_2}(dX)) \\
&= -\frac{6}{7}\phi(d\big[\pi_{\mathfrak{g}_2}(dX)\big]) \\
&= -\frac{1}{7} * (d\big[\pi_{\mathfrak{g}_2}(dX)\big] \wedge *\phi) \\
&= -\frac{3}{7}\pi_{\mathfrak{g}_2}(dX) \wedge *\pi_{\mathfrak{g}_2}(\delta\phi) \ .
\end{aligned}$$

Therefore, the upper sequence is a complex if and only if $(\delta\phi)_{\mathfrak{g}_2} = 0$. Fernández and Ugarte call this class of G_2-structures *integrable*, due to the analogy with almost-Hermitian structures. It is easy to check that that vertical arrows in the diagram constitute a cochain isomorphism if and only if $(\delta\phi)_V = 0$. Notice that this fits with what we know about the obstructions to the two sequences being complexes. The greater generality of the upper sequence makes it more useful than the lower one; in fact, the lower sequence is not even considered by Fernández and Ugarte. The differences between these two sequences do not appear to have been acknowledged in the literature.

For integrable G_2-structures the upper sequence is used in [FU98] to define the *canonical G_2-cohomology*. There is a short exact sequence of cochain complexes

$$0 \to \mathcal{B}^*(\mathcal{M}) \to deR^*(\mathcal{M}) \to \mathcal{A}^*(\mathcal{M}) \to 0$$

where $deR^*(\mathcal{M})$ is the usual de Rham complex, $\mathcal{A}^*(\mathcal{M})$ is our upper sequence and $\mathcal{B}^*(\mathcal{M})$ is a particular subcomplex of $deR^*(\mathcal{M})$. Fernández and Ugarte use this to construct a long exact sequence in cohomology which produces stronger lower bounds on the Betti numbers of manifolds with G_2 holonomy than those afforded by the de Rham complex alone. The excellent book by Joyce [Joy07] includes these results, as well as several other important results on the

topology of manifolds with exceptional holonomy.
With respect to the decomposition of the real spinor bundle $S\mathcal{M} = (\mathcal{M} \times \mathbb{R}) \oplus T\mathcal{M}$ we may write the Dirac operator $\mathcal{D}$ as

$$\mathcal{D} = \begin{pmatrix} 0 & -div \\ grad & curl \end{pmatrix} .$$

It is interesting to ask how harmonicity of vector and spinor fields behaves with respect to this decomposition. It is not difficult to show the following.

Theorem C.0.36. *Let $\mathcal{M}$ be a Riemannian manifold with G_2 holonomy, and $\mathcal{U} \subset \mathcal{M}$ a simply connected open subset. A spinor field can be written as (f, X) where f is a function and X is a vector field. Then,*

$$(f, X) \text{ is harmonic on } \mathcal{U} \;\Rightarrow\; X \text{ is harmonic on } \mathcal{U}$$

and

$$X \text{ is harmonic on } \mathcal{U} \;\Rightarrow\; \text{there exists an } f \text{ such that } (f, X) \text{ is harmonic on } \mathcal{U} .$$

Note that this theorem uses the full torsion-free hypothesis. We shall prove a similar result for manifolds with $Spin_7$ holonomy in the proof of Theorem E.0.45, but we do not state it separately.

Appendix D

Torsion of a G-structure

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let V be a representation of G. The map $\alpha : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^*$ is given by applying antisymmetrisation in the final two arguments to the subspace $\mathfrak{g} \otimes V^* \subset V \otimes V^* \otimes V^*$, included using the representation of G (which we shall omit). It is well-known that the *torsion sequence*

$$0 \to \mathfrak{g}^{(1)} \to \mathfrak{g} \otimes V^* \xrightarrow{\alpha} V \otimes \Lambda^2 V^* \to C \to 0$$

is an exact sequence of G-representations, where $\mathfrak{g}^{(1)} \stackrel{\text{def}}{=} \ker \alpha$ is the *1st prolongation algebra*[1] of $\mathfrak{g}$ with respect to V and C is the cokernel of α:

$$C = V \otimes \Lambda^2 V^* / \alpha(\mathfrak{g} \otimes V^*) \ .$$

The space C will be named the *space of intrinsic torsion*.

It is not difficult to show that $\mathfrak{g}^{(1)}$ is the intersection of two subrepresentations of $V \otimes V^* \otimes V^*$:

$$\mathfrak{g}^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2 V^* \subset V \otimes V^* \otimes V^* \ .$$

Now it is obvious how G acts on the algebra $\mathfrak{g}^{(1)}$. This fact is also very useful for calculating prolongations explicitly; it is a trivial consequence that $\mathfrak{so}_n^{(1)} = 0$.

It is also simple to show that the conformal algebra prolongs to $\mathfrak{co}_n^{(1)} = V^*$ and again to $\mathfrak{co}_n^{(2)} = 0$, demonstrating the existence and uniqueness of the normal conformal connection on the once-prolonged conformal frame bundle. We may also show that the symplectic prolongation series is $\mathfrak{sp}^{(i)} = S^{i+1} V^*$, which never terminates. If the prolongation series terminates we say $\mathfrak{g}$ (and the corresponding G-structure) has *finite type*, and it is a theorem of Kobayashi (see [Ste64]) that the automorphism group of a G-structure of finite type is itself a Lie group. This offers deeper insight into the difference between symplectic topology and the more tractable Cartan geometries.

If B_G is a G-structure over a manifold $\mathcal{M}^n$ and V is the representation such that $B_G \times_G V = T\mathcal{M}$, we get the corresponding *torsion exact sequence of associated bundles*[2]

$$\mathcal{M} \to B_G \times_G \mathfrak{g}^{(1)} \to B_G \times_G (\mathfrak{g} \otimes V^*) \xrightarrow{\alpha} B_G \times_G (V \otimes \Lambda^2 V^*) \to B_G \times_G C \to \mathcal{M}$$

[1] See [Ste64] for a good description of prolongation, both algebraically and for G-structures.

[2] Sternberg's excellent book [Ste64] does not use the language of associated vector bundles but instead uses equivariant, representation-valued functions on B_G.

where $\mathcal{M} \to B_G \times_G \mathfrak{g}^{(1)}$ is the zero section and $B_G \times_G C \to \mathcal{M}$ is the bundle projection. We can write this more succinctly as

$$\mathcal{M} \to Ad^{(1)}_{B_G} \to Ad_{B_G} \otimes T^*\mathcal{M} \xrightarrow{\alpha} T\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M} \to B_G \times_G C \to \mathcal{M} \ .$$

Two covariant derivative operators on $T\mathcal{M}$ induced by connections on the G-structure B_G differ by a section of $Ad_{B_G} \otimes T^*\mathcal{M}$, call it $\mathcal{A}$. A quick calculation shows that the torsion tensor fields of these operators differ by $2\alpha(\mathcal{A})$. Therefore, the images in C of their torsion tensors are equal. This image is called the *intrinsic torsion* of B_G. It is immediate from the torsion exact sequence that

Theorem D.0.37. *B_G admits a unique torsion-free connection if and only if $\mathfrak{g}^{(1)} = 0$ and $C = 0$.*

The Levi-Civita Theorem follows from the observation that $\mathfrak{so}_n^{(1)} = 0$.
The class of geometries such that $\mathfrak{g}^{(1)} \neq 0$ and $C = 0$ are those for which there exist many adapted torsion-free connections. In fact, such a structure admits many connections of any prescribed torsion in $T\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M}$, and the space of connections of a fixed torsion is a torsor for $\Gamma(Ad^{(1)}_{B_G})$ acting by translation. We are interested in the dual class—those geometries such that $\mathfrak{g}^{(1)} = 0$ and $C \neq 0$. In this case α is injective, which means that for any attainable torsion there is a unique connection attaining it. Many torsions are not attained. This class of geometries contains all *Riemannian G-structures*—those structures for which $G < SO_n$. For these, if a torsion-free connection exists it must be the restriction of the Levi-Civita connection and the G-structure must be parallel. This implies the holonomy group is contained in G.
For a Riemannian G-structure ($G < SO_n$, $V = \mathbb{R}^n$) the map α is injective and therefore $\mathfrak{g} \otimes V^*$ can be considered a subrepresentation of $V \otimes \Lambda^2 V^* \cong V \otimes \mathfrak{so}_n$. Then the space of intrinsic torsion C can be identified with the orthogonal complement of $\mathfrak{g} \otimes V^* \subset V \otimes \mathfrak{so}_n$, which is $\mathfrak{g}^\perp \otimes V^*$.

Proposition D.0.38. *For a Riemannian G-structure B_G, the intrinsic torsion is a section of*

$$B_G \times_G \mathfrak{g}^\perp \otimes V^* = T^*\mathcal{M} \otimes Ad^\perp_{B_G} \ .$$

Proof. Clear. □

To be consistent with the general notion of intrinsic torsion, throughout this section we have been distinguishing between V and its dual V^*. For Riemannian G-structures there is no need to do this and often we will consider the torsion to have values in $T\mathcal{M} \otimes Ad^\perp_{B_G}$. The following theorem can be found in [Sal89].

Theorem D.0.39. *Let B_G be a Riemannian G-structure over $\mathcal{M}$ and let η_0 be a G-invariant element of an SO_n-representation W. If ∇ is the covariant derivative on $SO(\mathcal{M}) \times_{SO_n} W$ induced by the Levi-Civita connection, then the intrinsic torsion of B_G may be identified with the derivative $\nabla\eta$ of the section η corresponding to η_0.*

Proof. Since W is a representation of SO_n and $G < SO_n$, it is also a representation of G. Therefore, the associated bundle $SO(\mathcal{M}) \times_{SO_n} W$ is isomorphic to the one formed using G, $B_G \times_G W \stackrel{\text{def}}{=} \mathcal{W}$. Choose a connection on B_G and extend it to a connection on $SO(\mathcal{M})$. This will agree with the Levi-Civita connection if and only if B_G is parallel, otherwise it will have torsion. It induces a covariant derivative operator ∇^G on $\mathcal{W}$, and $\nabla^G \eta = 0$. We can write

$$\nabla - \nabla^G = \mathcal{A}$$

for $\mathcal{A}$ a section of $Ad_{SO(\mathcal{M})} \otimes T^*\mathcal{M}$. The torsion exact sequence of $SO(\mathcal{M})$ is

$$\mathcal{M} \to Ad_{SO(\mathcal{M})} \otimes T^*\mathcal{M} \xrightarrow{\alpha} T\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M} \to \mathcal{M} \ .$$

The torsion of ∇^G is $2\alpha(\mathcal{A})$. Restriction of the isomorphism α to $Ad_{B_G} \otimes T^*\mathcal{M} \subset Ad_{SO(\mathcal{M})} \otimes T^*\mathcal{M}$ gives a subbundle $\alpha(Ad_{B_G} \otimes T^*\mathcal{M})$ of $T\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M}$. The intrinsic torsion of B_G is therefore

$$2\alpha(\mathcal{A}) \quad mod \quad \alpha(Ad_{B_G} \otimes T^*\mathcal{M}) \ .$$

Since α is injective, this can be identified with $2\mathcal{A}$ *mod* $Ad_{B_G} \otimes T^*\mathcal{M}$. The element η_0 is G-invariant so any section of $Ad_{B_G} \otimes T^*\mathcal{M}$ annihilates η. Thus, the equivalence class $2\mathcal{A}$ *mod* $Ad_{B_G} \otimes T^*\mathcal{M}$ can be identified with $2\mathcal{A}\eta$, which is a section of $\mathcal{W} \otimes T^*\mathcal{M}$. But $\nabla\eta = \mathcal{A}\eta$, so the intrinsic torsion can be identified with $2\nabla\eta$. We ignore the factor of two since it makes no qualitative difference, so we may consider the intrinsic torsion to be $\nabla\eta$. □

Remark D.0.40. *We can use a very similar proof to show that if W is not a representation of SO_n but instead of $Spin_n$, the result still holds. However, in this case we require that $G < Spin_n$, and since $G < SO_n$ by hypothesis, this modification will work when G lifts isomorphically to $Spin_n$ by the standard double cover $Spin_n \to SO_n$, i.e. when G is simply connected. Both G_2 and $Spin_7^+$ are simply connected.*

We end this appendix with an aside specific to G_2-structures. If we have a G_2-structure B_{G_2} over a 7-dimensional manifold $\mathcal{M}$ then we can consider the restriction κ of the Levi-Civita connection form to B_{G_2}, which is a connection form on B_{G_2} if and only if $\mathcal{M}$ has G_2 holonomy. In general, the kernel of κ is $TB_{G_2} \cap \mathscr{H}^{LC}$ where $\mathscr{H}^{LC}$ is the distribution of the Levi-Civita connection. Therefore κ is a Cartan connection on B_{G_2} if and only if $TB_{G_2} \cap \mathscr{H}^{LC} = 0$. It turns out that this condition is not best described using the *type* of the G_2-structure, but rather

Theorem D.0.41. *Let B_{G_2} be a G_2-structure whose canonical spinor field φ satisfies*

$$\nabla_X \varphi = A_X \cdot \varphi$$

for all vector fields X and for A a section of $B_{G_2} \times_{G_2} V \otimes V \cong \mathrm{End}T\mathcal{M}$. Then the restriction κ of the Levi-Civita connection form to $B_{G_2} \subset SO(\mathcal{M})$ is a Cartan connection if and only if A is non-degenerate.

We omit the proof. It follows immediately that for nearly-parallel G_2-structures, A is non-degenerate so κ is a Cartan connection. For the other types the situation is less clear: are there G_2-structures with $\nabla_X \varphi = A_X \cdot \varphi$ such that A is non-zero yet degenerate?

Since the G_2-structure is Riemannian the simplest tractor bundle of this Cartan connection is just the tangent bundle, so is not anything new. The Cartan connection may provide a nice perspective nonetheless.

Note that there is no chance of an analogous theorem for $Spin_7^+$-structures because the restriction of the Levi-Civita connection to a $Spin_7^+$-structure $B_{Spin_7^+} \subset SO(\mathcal{M})$ takes values in $\mathfrak{so}_8$, in which $\mathfrak{spin}_7^+$ has codimension seven, rather than the required eight.

Appendix E

Group Actions and Holonomy

There are topological restrictions on compact spin manifolds admitting group actions. We shall expose some of these using index theory, although we make no attempt to explain the index theory itself; a good reference for this is [Roe98]. We assume the group acts by isometries and preserves the spin structure. Throughout, G will denote a compact connected Lie group.

Definition E.0.42. *A G-action on a Riemannian spin manifold $\mathcal{M}$ induces a G-action on $SO(\mathcal{M})$. The action is called **spin structure-preserving** if this lifts to a G-action on $Spin(\mathcal{M})$, i.e. if there is a map so that*

$$
\begin{tikzcd}
 & Spin(\mathcal{M}) \arrow[d] \\
 & SO(\mathcal{M}) \arrow[d] \\
G \arrow[r] \arrow[ur] \arrow[uur] & \mathcal{M}
\end{tikzcd}
$$

commutes. See Definition 10.2.7 for comparison.

Atiyah and Hirzebruch proved

Theorem E.0.43. *([AH70]) Let $\mathcal{M}$ be a compact even-dimensional Riemannian spin manifold admitting a non-trivial spin structure-preserving G-action. Then*

$$\hat{A}(\mathcal{M}) = 0 \ .$$

See [Roe98] for the theory behind the genus $\hat{A}(\mathcal{M})$. We shall also need

Theorem E.0.44. *([Joy07]) Let $\mathcal{M}$ be a compact Riemannian manifold with holonomy $Spin_7$. Then $\mathcal{M}$ is simply connected.*

The following theorem is well-known.

Theorem E.0.45. *Let $\mathcal{M}$ be a compact Riemannian manifold with holonomy $Spin_7$. Then $\mathcal{M}$ does not admit a non-trivial G-action.*

Proof. First note that since $\mathcal{M}$ is simply connected by Theorem E.0.44, any G-action is spin structure-preserving. As usual, denote the real spinor bundle of $\mathcal{M}$ by $S\mathcal{M}$, so that

$$S\mathcal{M} = S\mathcal{M}^+ \oplus S\mathcal{M}^- = \mathbb{R}\Psi \oplus \Psi^\perp \oplus T\mathcal{M}$$

as a $Spin_7$-associated bundle. We have used the fact that $S\mathcal{M}^- = T\mathcal{M}$ in this setting. There are two Weitzenböck formulae relevant to us:

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa \ , \quad 2\Delta = \nabla^*\nabla + Ric$$

where $\mathcal{D}$ is the Dirac operator, κ is the scalar curvature, Δ is the form Laplacian (on 1-forms here) and Ric is the Ricci endomorphism. Since the holonomy is $Spin_7$, $Ric = 0$ and $\kappa = 0$ and so with respect to the identification of $S\mathcal{M}^-$ with $T\mathcal{M}$,

$$\mathcal{D}^2 = 2\Delta \ .$$

The splitting $S\mathcal{M} = S\mathcal{M}^+ \oplus S\mathcal{M}^-$ gives us a splitting of the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$$

so that for a spinor field $(\chi_1, \chi_2) \in \Gamma(S\mathcal{M}^+ \oplus S\mathcal{M}^-)$

$$\mathcal{D} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_-\chi_2 \\ \mathcal{D}_+\chi_1 \end{pmatrix}$$

and

$$\mathcal{D}^2 \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_-\mathcal{D}_+\chi_1 \\ \mathcal{D}_+\mathcal{D}_-\chi_2 \end{pmatrix} \ .$$

Suppose $(0, \zeta)$ is a spinor field in the kernel of $\mathcal{D}$, i.e. $\mathcal{D}_-\zeta = 0$. It follows that $\mathcal{D}^2\zeta = 0$ and therefore $\Delta\zeta = 0$, so ζ is a harmonic vector field on $\mathcal{M}$. Conversely, if ζ is a harmonic vector field on $\mathcal{M}$, $\mathcal{D}^2\zeta = \mathcal{D}_+\mathcal{D}_-\zeta = 0$. It is a fact that $\ker \mathcal{D}^2 = \ker \mathcal{D}$ (see [LM89]) and so $\mathcal{D}_-\zeta = 0$. This shows that there is a one-to-one correspondence between elements of $\ker \mathcal{D}_-$ and harmonic vector fields on $\mathcal{M}$. But $\mathcal{M}$ is simply connected, so the Hodge Theorem implies

$$\ker \mathcal{D}_- = 0 \ .$$

On the other hand, the canonical spinor field Ψ is parallel and therefore also harmonic; $\mathcal{D}_+\Psi = 0$. So

$$\dim \ker \mathcal{D}_+ > 0 \ .$$

The index of the Dirac operator must satisfy

$$Ind(\mathcal{D}) \stackrel{\text{def}}{=} \dim \ker \mathcal{D}_+ - \dim \ker \mathcal{D}_- = \dim \ker \mathcal{D}_+ > 0 \ .$$

Using the Atiyah-Singer Index Theorem

$$\hat{A}(\mathcal{M}) = Ind(\mathcal{D}) > 0 \ .$$

Theorem E.0.43 then implies no such group action can exist. □

Whilst we have shown that $\hat{A}(\mathcal{M}) > 0$, Joyce [Joy07] goes much further and proves that $\hat{A}(\mathcal{M})$ must be 1, 2, 3 or 4. The same book also has excellent results on the refined Betti numbers of manifolds with holonomy G_2 or $Spin_7$, as mentioned in Appendix C.

One corollary of Theorem E.0.45 is that a compact simply connected Riemannian manifold with holonomy $Spin_7$ must have a discrete isometry group. We should not be too impressed with this particular fact though, because it can be more easily proven using less advanced technology. If X is a Killing vector field on a compact Riemannian manifold $\mathcal{M}$ then it is easy to show that $\langle \nabla^* \nabla X, Y \rangle = \langle \Delta X, Y \rangle$ for any vector field Y, where $\langle \cdot, \cdot \rangle$ is the inner product of vector fields induced by the metric. Acting on 1-forms we have the Weitzenböck formula $2\Delta = \nabla^* \nabla + Ric$ which we use to show $\nabla^* \nabla X = Ric(X)$, and taking the inner product with X gives $||\nabla X||^2 = \langle Ric(X), X \rangle$. Therefore, a Killing vector field on a compact Ricci-flat manifold is parallel. If $\mathcal{M}$ has holonomy equal to $Spin_7$ then it has no parallel vector fields and therefore no Killing vector fields, and its isometry group must be discrete. The same argument does work for G_2 holonomy this time.

The hypotheses of Theorem E.0.45 do not refer to an *isometric* G-action, as Theorem E.0.43 does not. Therefore, the result is true for any compact manifold which merely admits a metric with holonomy $Spin_7$.

Theorem E.0.45 holds also for 8-dimensional compact simply connected Riemannian manifolds with holonomy SU_4 or Sp_2. In the Calabi-Yau case there are two canonical parallel spinor fields which have the same chirality because 4 is even. In the hyper-Kähler case there are three such fields all of the same chirality (see Wang's classic paper [Wan89] for these facts). Since the index of a Dirac complex vanishes in odd-dimensions we do not have a similar statement for manifolds with holonomy G_2.

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